

# CATEGORICAL RESOLUTIONS OF BOUNDED DERIVED CATEGORIES

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**ABSTRACT.** Using a relative version of Auslander's formula, we show that bounded derived category of every artin algebra admits a categorical resolution. This, in particular, implies that bounded derived categories of artin algebras of finite global dimension determine bounded derived categories of all artin algebras. This in a sense provides a categorical level of Auslander's result stating that artin algebras of finite global dimension determine all artin algebras.

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## 1. INTRODUCTION

Let  $X$  be an algebraic variety. A resolution of singularities of  $X$  is a certain (proper and birational) morphism  $\pi : \tilde{X} \rightarrow X$ , where  $\tilde{X}$  is a non-singular algebraic variety. The functor  $\mathbb{D}^b(\text{coh}\tilde{X}) \rightarrow \mathbb{D}^b(\text{coh}X)$  induced by  $\pi$  enjoys some remarkable properties. The bounded derived categories of coherent sheaves on  $\tilde{X}$  and  $X$  are related by two natural functors, known as the derived pushforward  $\pi_* : \mathbb{D}^b(\text{coh}\tilde{X}) \rightarrow \mathbb{D}^b(\text{coh}X)$  and the derived pullback functor  $\pi^* : \mathbb{D}^{\text{perf}}(\text{coh}X) \rightarrow \mathbb{D}^b(\text{coh}\tilde{X})$ , such that  $\pi^*$  is left adjoint to  $\pi_*$ . Here  $\mathbb{D}^{\text{perf}}(\text{coh}X)$  stands for the full subcategory of  $\mathbb{D}^b(\text{coh}X)$  consisting of perfect complexes. If furthermore,  $X$  have

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rational singularities, then the unit of adjunction  $1_{\mathbb{D}^{\text{perf}}} \rightarrow \pi_*\pi^*$  is an isomorphism and  $\pi_*$  induces an identification between  $\mathbb{D}^b(\text{coh}X)$  and the quotient of  $\mathbb{D}^b(\text{coh}\tilde{X})$  by the kernel of  $\pi_*$ .

Based on this observation, Kuznetsov [Ku] introduced the notion of a categorical resolution of singularities. By definition a categorical resolution of  $\mathbb{D}^b(\mathcal{A})$  is a smooth triangulated category  $\tilde{\mathbb{D}}$  and a pair of functors  $\pi_*$  and  $\pi^*$  satisfying almost similar conditions as above, see [Ku, Definition 3.2]. Recall that a triangulated category  $\mathfrak{T}$  is called smooth if it is triangle equivalent to the bounded derived category of an abelian category  $\mathcal{A}$  with vanishing singularity category, i.e.  $\mathbb{D}_{\text{sg}}^b(\mathcal{A}) = 0$ .

Note that Bondal and Orlov [BO] also took the above observation as a template and defined a categorical desingularization of a triangulated category  $\mathfrak{T}$  to be a pair  $(\mathcal{A}, \mathcal{K})$ , where  $\mathcal{A}$  is an abelian category of finite homological dimension and  $\mathcal{K}$  is a thick triangulated subcategory of  $\mathbb{D}^b(\mathcal{A})$  such that  $\mathfrak{T} \simeq \mathbb{D}^b(\mathcal{A})/\mathcal{K}$ , see [BO, §5]. Recall that a full triangulated subcategory of a triangulated category  $\mathfrak{T}$  is called thick if it is closed under taking direct summands.

Recently, Zhang [Z] combined these two categorical levels of the notion of a resolution of singularities and suggested a new definition for a categorical resolution of a non-smooth triangulated category [Z, Definition 1.1]. He then proved that if  $\Lambda$  is an artin algebra of infinite global dimension and has a module  $T$  with  $\text{id}_{\Lambda}T < 1$  such that  ${}^{\perp}T$  is of finite type, then the bounded derived category  $\mathbb{D}^b(\text{mod-}\Lambda)$  admits a categorical resolution [Z, Theorem 4.1]. The main technique for proving this result is the notion of relative derived categories studied by several authors in different settings, see e.g [N], [Bu] and [GZ].

In this paper, we generalize Zhang's result to arbitrary artin algebras and show that the bounded derived category of every artin algebra admits a categorical resolution. The technique for the proof is based on a relative version of the so-called Auslander's Formula [Au1] and [L]. This relative version will be treated explicitly in Section 3 and is of independent interest. Auslander's formula suggests that for studying an abelian category  $\mathcal{A}$  one may study  $\text{mod-}\mathcal{A}$ , the category of finitely presented additive functors on  $\mathcal{A}$ , that has nicer homological properties than  $\mathcal{A}$ , and then translate the results back to  $\mathcal{A}$ . Here, among other results, we replace  $\mathcal{A}$  with a contravariantly finite subcategory  $\mathcal{X}$  of  $\mathcal{A}$  that contains projective objects. We establish the existence of a recollement and show that Auslander's formula is in fact derived from this recollement, see Theorem 3.7. Similar result will be provided for finitely presented covariant functors, Theorem 3.8. Beside Auslander's formula, some interesting corollaries will be derived from this recollement, among them Auslander's four terms exact sequence. Some examples and also applications will be presented in Section 4. Then we consider the category of left  $\Lambda$ -modules, where  $\Lambda$  is an artin algebra, in Section 5 and present a recollement containing  $\Lambda\text{-mod}$ , see Theorem 5.2.4 below. To this end we discuss briefly the structure of injective finitely presented covariant functors in a subsection, Subsection 5.1. Using this, for every resolving contravariantly finite subcategory  $\mathcal{X}$  of  $\text{mod-}\Lambda$ , we construct a duality  $\tilde{D}_{\mathcal{X}}$  between the categories of right and left  $\Lambda$ -modules, also in stable level.

Last section is devoted to the proof of our main theorem. To this end, besides Auslander's formula we use the following known result of Auslander. In his Queen Mary College lectures [Au2] he proves that for an artin algebra  $\Lambda$  there exists an artin algebra  $\tilde{\Lambda}$  of finite global dimension and an idempotent  $e$  of  $\tilde{\Lambda}$  such that  $\Lambda = e\tilde{\Lambda}e$ . Hence, as he mentioned, artin algebras of finite global dimension determine all artin algebras. Later on Dlab and Ringel [DR] showed that  $\tilde{\Lambda}$  in Auslander's construction is in fact a quasi-hereditary artin algebra. Our volunteer for the proof of the main theorem is  $\tilde{\Lambda}$ , that throughout for ease of reference we call it A-algebra of  $\Lambda$ . 'A' stands both for 'Auslander' and also 'Associated' algebra.

Our main theorem, in particular, implies that for any artin algebra  $\Lambda$  of infinite global dimension there exists an artin algebra  $\tilde{\Lambda}$  of finite global dimension, its  $\Lambda$ -algebra, such that  $\mathbb{D}^b(\text{mod-}\Lambda)$  is equivalent to a quotient of  $\mathbb{D}^b(\text{mod-}\tilde{\Lambda})$ . Therefore artin algebras of finite global dimensions, or better quasi-hereditary algebras, determine all artin algebras also in the categorical level.

## 2. PRELIMINARIES

Let  $\mathcal{A}$  be an additive skeletally small category. The Hom sets will be denoted either by  $\text{Hom}_{\mathcal{A}}(-, -)$ ,  $\mathcal{A}(-, -)$  or even just  $(-, -)$ , if there is no risk of ambiguity. Let  $\mathcal{X}$  be a full subcategory of  $\mathcal{A}$ . By definition, a (right)  $\mathcal{X}$ -module is a contravariant additive functor  $F : \mathcal{X} \rightarrow \mathcal{A}b$ , where  $\mathcal{A}b$  denotes the category of abelian groups. The  $\mathcal{X}$ -modules and natural transformations between them, called morphisms, form an abelian category denoted by  $\text{Mod-}\mathcal{X}$  or sometimes  $(\mathcal{X}^{\text{op}}, \mathcal{A}b)$ . An  $\mathcal{X}$ -module  $F$  is called finitely presented if there exists an exact sequence

$$\mathcal{X}(-, X) \rightarrow \mathcal{X}(-, X') \rightarrow F \rightarrow 0,$$

with  $X$  and  $X'$  in  $\mathcal{X}$ . All finitely presented  $\mathcal{X}$ -modules form a full subcategory of  $\text{Mod-}\mathcal{X}$ , denoted by  $\text{mod-}\mathcal{X}$  or sometimes f.p.  $(\mathcal{C}^{\text{op}}, \mathcal{A}b)$ . Covariant additive functors and its full subcategory consisting of finitely presented (left)  $\mathcal{X}$ -modules will be denoted by  $\mathcal{X}\text{-Mod}$  and  $\mathcal{X}\text{-mod}$ , respectively. Since every finitely generated subobject of a finitely presented object is finitely presented [Au1, Page 200], Auslander called them coherent functors.

The Yoneda embedding  $\mathcal{X} \hookrightarrow \text{mod-}\mathcal{X}$ , sending each  $X \in \mathcal{X}$  to  $\mathcal{X}(-, X) := \mathcal{A}(-, X)|_{\mathcal{X}}$ , is a fully faithful functor. Note that for each  $X \in \mathcal{X}$ ,  $\mathcal{X}(-, X)$  is a projective object of  $\text{mod-}\mathcal{X}$ . Moreover, every projective objects of  $\text{mod-}\mathcal{X}$  is a direct summand of  $\mathcal{X}(-, X)$ , for some  $X \in \mathcal{X}$ . These facts are known and also easy to prove using Yoneda Lemma.

A morphism  $X \rightarrow Y$  is a weak kernel of a morphism  $Y \rightarrow Z$  in  $\mathcal{X}$  if the induced sequence

$$(-, X) \longrightarrow (-, Y) \longrightarrow (-, Z)$$

is exact on  $\mathcal{X}$ . It is known that  $\text{mod-}\mathcal{X}$  is an abelian category if and only if  $\mathcal{X}$  admits weak kernels, see e.g. [Au2, Chapter III, §2] or [Kr2, Lemma 2.1].

Let  $A \in \mathcal{A}$ . A morphism  $\varphi : X \rightarrow A$  with  $X \in \mathcal{X}$  is called a right  $\mathcal{X}$ -approximation of  $A$  if  $\mathcal{A}(-, X)|_{\mathcal{X}} \longrightarrow \mathcal{A}(-, A)|_{\mathcal{X}} \longrightarrow 0$  is exact, where  $\mathcal{A}(-, A)|_{\mathcal{X}}$  is the functor  $\mathcal{A}(-, A)$  restricted to  $\mathcal{X}$ . Hence  $A$  has a right  $\mathcal{X}$ -approximation if and only if  $(-, A)|_{\mathcal{X}}$  is a finitely generated objects of  $\text{Mod-}\mathcal{X}$ .  $\mathcal{X}$  is called contravariantly finite if every object of  $\mathcal{A}$  admits a right  $\mathcal{X}$ -approximation. Dually, one can define the notion of left  $\mathcal{X}$ -approximations and covariantly finite subcategories.  $\mathcal{X}$  is called functorially finite, if it is both covariantly and contravariantly finite.

It is obvious that if  $\mathcal{X}$  is contravariantly finite, then it admits weak kernels and hence  $\text{mod-}\mathcal{X}$  is an abelian category.

**2.1. Recollements of abelian categories.** A subcategory  $\mathcal{S}$  of an abelian category  $\mathcal{A}$  is called a Serre subcategory, if it is closed under taking subobjects, quotients and extensions. Let  $\mathcal{S}$  be a Serre subcategory of  $\mathcal{A}$ . The quotient category  $\mathcal{A}/\mathcal{S}$  is by definition the localization of  $\mathcal{A}$  with respect to the collection of morphisms that their kernels and cokernels are in  $\mathcal{S}$ . It is known [Ga] that  $\mathcal{A}/\mathcal{S}$  is an abelian category and the quotient functor  $Q : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{S}$  is exact with  $\text{Ker}Q = \mathcal{S}$ .

Let  $\mathcal{A}'$ ,  $\mathcal{A}$  and  $\mathcal{A}''$  be abelian categories. A recollement of  $\mathcal{A}$  with respect to  $\mathcal{A}'$  and  $\mathcal{A}''$  is a diagram

$$\begin{array}{ccccc} & u_\lambda & & v_\lambda & \\ \mathcal{A}' & \xleftarrow{\quad} & \mathcal{A} & \xleftarrow{\quad} & \mathcal{A}'' \\ & u_\rho & & v_\rho & \end{array}$$

$\begin{array}{c} \xrightarrow{u} \\ \xleftarrow{v} \end{array}$

of additive functors such that  $u$ ,  $v_\lambda$  and  $v_\rho$  are fully faithful,  $(u_\lambda, u)$ ,  $(u, u_\rho)$ ,  $(v_\lambda, v)$  and  $(v, v_\rho)$  are adjoint pairs and  $\text{Im} u = \text{Ker} v$ .

Note that  $v_\lambda$  is fully faithful if and only if  $v_\rho$  is fully faithful. It follows quickly in a recollement situation that the functors  $u$  and  $v$  are exacts,  $u$  induces an equivalence between  $\mathcal{A}'$  and the Serre subcategory  $\text{Im} u = \text{Ker} v$  of  $\mathcal{A}$  and there exists an equivalence  $\mathcal{A}'' \simeq \mathcal{A}/\mathcal{A}'$ , see for instance [Ps, Remark 2.2].

A localisation, resp. colocalisation, sequence consists only the lower, resp. upper, two rows of a recollement such that the functors appearing in them satisfy all the conditions of a recollement that involve only these functors.

Two recollements  $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$  and  $(\mathcal{B}', \mathcal{B}, \mathcal{B}'')$  are equivalent if there exist equivalences  $\Phi : \mathcal{A}' \rightarrow \mathcal{B}'$ ,  $\Psi : \mathcal{A} \rightarrow \mathcal{B}$  and  $\Theta : \mathcal{A}'' \rightarrow \mathcal{B}''$ , such that the six diagrams associated to the six functors of the recollements commute up to natural equivalences, see [PV, Lemma 4.2].

**Remark 2.1.** Let  $v : \mathcal{A} \rightarrow \mathcal{A}''$  be an exact functor between abelian categories admitting a left and a right adjoint,  $v_\lambda$ ,  $v_\rho$ , respectively, such that one of the  $v_\lambda$  or  $v_\rho$ , and hence both of them, are fully faithful. Then we get a recollement  $(\text{Ker} v, \mathcal{A}, \mathcal{A}'')$  of abelian categories, see [Ps, Remark 2.3] for details.

In our recollement  $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$  let us denote the counits of the adjunctions  $uu_\rho \rightarrow 1_{\mathcal{A}}$  and  $v_\lambda v \rightarrow 1_{\mathcal{A}}$  by  $\eta^{uu_\rho}$  and  $\eta^{v_\lambda v}$ , respectively and the units of the adjunctions  $1_{\mathcal{A}} \rightarrow uu_\lambda$  and  $1_{\mathcal{A}} \rightarrow v_\rho v$  by  $\delta^{uu_\lambda}$  and  $\delta^{v_\rho v}$ , respectively.

**Remark 2.2.** Let  $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$  be a recollement of abelian categories. Then for any  $A \in \mathcal{A}$  there exists the following two exact sequences

$$\begin{aligned} 0 \longrightarrow uu_\rho(A) \xrightarrow{\eta_A^{uu_\rho}} A \xrightarrow{\delta_A^{v_\rho v}} v_\rho v(A) \longrightarrow \text{Coker} \delta_A^{v_\rho v} \longrightarrow 0; \\ 0 \longrightarrow \text{Ker} \eta_A^{v_\lambda v} \longrightarrow v_\lambda v(A) \xrightarrow{\eta_A^{v_\lambda v}} A \xrightarrow{\delta_A^{uu_\lambda}} uu_\lambda(A) \longrightarrow 0. \end{aligned}$$

Moreover, there exist  $A'_0$  and  $A'_1 \in \mathcal{A}'$  such that  $\text{Coker} \delta_A^{v_\rho v} = u(A'_0)$  and  $\text{Ker} \eta_A^{v_\lambda v} = u(A'_1)$ . For the proof see [FP] and [PV, Proposition 2.8].

**2.2. Dualising  $R$ -varieties.** Let  $R$  be a commutative artinian ring. The notion of dualising  $R$ -varieties is introduced by Auslander and Reiten [AR1]. A dualising  $R$ -variety can be considered as an analogue of the category of finitely generated projective modules over an artin algebra, but with possibly infinitely many indecomposable objects, up to isomorphisms. Let  $\mathcal{X}$  be an additive  $R$ -linear essentially small category.  $\mathcal{X}$  is called a dualising  $R$ -variety if the functor  $\text{Mod-}\mathcal{X} \rightarrow \text{Mod-}\mathcal{X}^{\text{op}}$  taking  $F$  to  $DF$ , induces a duality  $\text{mod-}\mathcal{X} \rightarrow \text{mod-}\mathcal{X}^{\text{op}}$ . Note that  $D(-) := \text{Hom}_R(-, E)$ , where  $E$  is the injective envelope of  $R/\text{rad} R$ . If  $\mathcal{X}$  is a dualising  $R$ -variety, then  $\text{mod-}\mathcal{X}$  and  $\text{mod-}\mathcal{X}^{\text{op}}$  are abelian categories with enough projectives and injectives [AR1, Theorem 2.4].

**Remark 2.3.** Let  $\mathcal{X}$  be a dualising  $R$ -variety. Then

- (i)  $\text{mod-}\mathcal{X}$  is a dualising  $R$ -variety [AR1, Proposition 2.6].

- (ii) Every functorially finite subcategory of  $\mathcal{X}$  is again a dualising  $R$ -variety [AS, Theorem 2.3], [II, Proposition 1.2].

**Remark 2.4.** The most basic example of a dualising  $R$ -variety is the category  $\text{prj-}\Lambda$ , finitely generated projective  $\Lambda$ -modules, where  $\Lambda$  is an artin algebra [AR1, Proposition 2.5]. Since  $\text{mod-}\Lambda \cong \text{mod-}(\text{prj-}\Lambda)$ , by the above remark,  $\text{mod-}\Lambda$  and also any functorially finite subcategory of it is dualising  $R$ -variety.

**2.3. Stable categories.** Let  $\mathcal{A}$  be an abelian category with enough projective objects. Let  $\mathcal{X}$  be a subcategory of  $\mathcal{A}$  containing  $\text{Prj-}\mathcal{A}$ , the full subcategory of  $\mathcal{A}$  consisting of projective objects. The stable category of  $\mathcal{X}$  denoted by  $\underline{\mathcal{X}}$  is a category whose objects are the same as those of  $\mathcal{X}$ , but the hom-set  $\underline{\mathcal{X}}(\underline{X}, \underline{X'})$  of  $X, X' \in \mathcal{X}$  is defined as  $\underline{\mathcal{X}}(\underline{X}, \underline{X'}) := \frac{\mathcal{A}(X, X')}{\mathcal{P}(X, X')}$ , where  $\mathcal{P}(X, X')$  consists of all morphisms from  $X$  to  $X'$  that factor through a projective object. We have the canonical functor  $\pi : \mathcal{X} \rightarrow \underline{\mathcal{X}}$ , defined by identity on objects but morphism  $f : X \rightarrow Y$  will be sent to the residue class  $\underline{f} := f + \mathcal{P}(X, X')$ .

Throughout the paper,  $\Lambda$  denotes an artin algebra over a commutative artinian ring  $R$ ,  $\text{Mod-}\Lambda$  denotes the category of all right  $\Lambda$ -modules and  $\text{mod-}\Lambda$  denotes its full subcategory consisting of all finitely presented modules. Moreover,  $\text{Prj-}\Lambda$ , resp.  $\text{prj-}\Lambda$ , denotes the full subcategory of  $\text{Mod-}\Lambda$ , resp.  $\text{mod-}\Lambda$ , consisting of projective, resp. finitely generated projective, modules. Similarly, the subcategories  $\text{Inj-}\Lambda$  and  $\text{inj-}\Lambda$  are defined.  $D(-) := \text{Hom}_R(-, E)$ , where  $E$  is the injective envelope of  $R/\text{rad}R$ , denotes the usual duality. For a  $\Lambda$ -module  $M$ , we let  $\text{add-}M$  denote the class of all modules that are isomorphic to a direct summand of a finite direct sum of copies of  $M$ .

### 3. RELATIVE AUSLANDER FORMULA

Let  $\mathcal{A}$  be an abelian category. Auslander's work on coherent functors [Au1, page 205] implies that the Yoneda functor  $\mathcal{A} \rightarrow \text{mod-}\mathcal{A}$  induces a localisation sequence of abelian categories

$$\text{mod}_0\text{-}\mathcal{A} \rightrightarrows \text{mod-}\mathcal{A} \rightrightarrows \mathcal{A}$$

where  $\text{mod}_0\text{-}\mathcal{A}$  is the full subcategory of  $\text{mod-}\mathcal{A}$  consisting of those functors  $F$  for them there exists a presentation  $\mathcal{A}(-, A) \rightarrow \mathcal{A}(-, A') \rightarrow F \rightarrow 0$  such that  $A \rightarrow A'$  is an epimorphism. See [Kr1, Theorem 2.2], where  $\text{mod}_0\text{-}\mathcal{A}$  is denoted by  $\text{eff}\mathcal{A}$ . This, in particular, implies that the functor  $\text{mod-}\mathcal{A} \rightarrow \mathcal{A}$ , that is the left adjoint of the Yoneda functor, induces an equivalence

$$\frac{\text{mod-}\mathcal{A}}{\text{mod}_0\text{-}\mathcal{A}} \simeq \mathcal{A}.$$

Following Lenzing [L] this equivalence will be called the Auslander's formula.

In this section, we show that for a right coherent ring  $A$  and every contravariantly finite subcategory  $\mathcal{X}$  of  $\text{mod-}A$  containing projective  $A$ -modules, there exists a recollement

$$\text{mod}_0\text{-}\mathcal{X} \rightrightarrows \text{mod-}\mathcal{X} \rightrightarrows \text{mod-}A$$

This, in particular, implies that

$$\frac{\text{mod-}\mathcal{X}}{\text{mod}_0\text{-}\mathcal{X}} \simeq \text{mod-}A.$$

In case we set  $\mathcal{X} = \text{mod-}A$ , we get the usual Auslander's formula. The importance will be illustrated by some interesting examples of  $\mathcal{X}$ , see Section 4.

Let us begin with the following proposition that is an immediate consequence of Proposition 2.1 of [Aul].

**Proposition 3.1.** *Let  $\mathcal{A}$  be an abelian category and  $\mathcal{X}$  be a full subcategory of  $\mathcal{A}$  that admits weak kernels. Consider the Yoneda embedding  $Y : \mathcal{X} \rightarrow \text{mod-}\mathcal{X}$ . Then given any abelian category  $\mathcal{D}$ , the induced functor  $Y^{\mathcal{D}} : (\text{mod-}\mathcal{X}, \mathcal{D}) \rightarrow (\mathcal{X}, \mathcal{D})$  has a left adjoint  $Y_{\mathcal{X}}^{\mathcal{D}}$  such that for each  $F \in (\mathcal{X}, \mathcal{D})$ ,  $Y_{\mathcal{X}}^{\mathcal{D}}(F)$  is right exact and  $Y_{\mathcal{X}}^{\mathcal{D}}(F)Y = F$ .*

*Proof.* Set  $\mathcal{A} = \text{Mod-}\mathcal{X}$  and  $\mathcal{P} = (-, \mathcal{X})$  in the settings of Proposition 2.1 of [Aul]. Then  $\mathcal{P}(\mathcal{A}) = \text{mod-}\mathcal{X}$ , which is an abelian category, because  $\mathcal{X}$  has weak kernels. So the result follows immediately.  $\square$

**Remark 3.2.** Consider the same settings as in the above proposition. Following Auslander [Aul], set  $\mathcal{D} := \mathcal{A}$  and let  $\ell : \mathcal{X} \rightarrow \mathcal{A}$  be the inclusion. Hence  $\ell$  can be extended to a right exact functor  $Y_{\mathcal{X}}^{\mathcal{A}}(\ell) : \text{mod-}\mathcal{X} \rightarrow \mathcal{A}$ . Let us for simplicity denote  $Y_{\mathcal{X}}^{\mathcal{A}}(\ell)$  by  $\vartheta$ .

We could provide an explicit interpretation of  $\vartheta$ . Let  $F \in \text{mod-}\mathcal{X}$  and  $\mathcal{X}(-, X_1) \xrightarrow{(-, d)} \mathcal{X}(-, X_0) \rightarrow F \rightarrow 0$  be a projective presentation of  $F$ , where  $X_0$  and  $X_1$  are in  $\mathcal{X}$ . Apply  $\vartheta$  and use the fact that by the above proposition  $\vartheta(\mathcal{X}(-, X)) = X$ , for all  $X$  in  $\mathcal{X}$ ,  $\vartheta(F)$  is then determined by the exact sequence

$$X_1 \xrightarrow{d} X_0 \rightarrow \vartheta(F) \rightarrow 0.$$

Moreover, if  $f : F \rightarrow F'$  is a morphism in  $\text{mod-}\mathcal{X}$ , then clearly it can be lifted to a morphism between their projective presentations. Yoneda lemma now come to play for the projective terms to provide unique morphisms on  $X_1$  and  $X_0$ . These morphisms then induce a morphism  $\vartheta(F) \rightarrow \vartheta(F')$ , which is exactly  $\vartheta(f)$ .

**Lemma 3.3.** *Let  $\mathcal{A}$  be an abelian category with enough projective objects and  $\mathcal{X}$  be a contravariantly finite subcategory of  $\mathcal{A}$  containing all projectives. Then  $\vartheta$  is an exact functor.*

*Proof.* Since  $\mathcal{X}$  is contravariantly finite, it admits weak kernels and hence  $\text{mod-}\mathcal{X}$  is an abelian category. Since  $F \in \text{mod-}\mathcal{X}$  and  $\mathcal{X}$  is contravariantly finite and contains projectives, there exists an exact sequence  $\mathcal{X}(-, X_2) \rightarrow \mathcal{X}(-, X_1) \rightarrow \mathcal{X}(-, X_0) \rightarrow F \rightarrow 0$  in  $\text{mod-}\mathcal{X}$  such that the induced sequence  $X_2 \rightarrow X_1 \rightarrow X_0$  is exact. So by applying  $\vartheta$ , we get the exact sequence

$$\vartheta(\mathcal{X}(-, X_2)) \rightarrow \vartheta(\mathcal{X}(-, X_1)) \rightarrow \vartheta(\mathcal{X}(-, X_0)).$$

So  $L_1\vartheta(F)$ , the first left derived functor of  $F$ , vanishes. Since this happens for all  $F \in \text{mod-}\mathcal{X}$ , we deduce that  $\vartheta$  is exact.  $\square$

Towards the end of this subsection, we show that if  $\mathcal{A} = \text{mod-}A$ , where  $A$  is a right coherent ring and if  $\mathcal{X}$  is a contravariantly finite subcategory of  $\text{mod-}A$  containing  $\text{prj-}A$ , then  $\vartheta$  has a left adjoint  $\vartheta_{\lambda}$  and a fully faithful right adjoint  $\vartheta_{\rho}$ . Let us first define the adjoint functors.

**3.4.** Let  $M \in \text{mod-}A$ . To define  $\vartheta_{\lambda}$ , let  $A^n \xrightarrow{d} A^m \xrightarrow{\varepsilon} M \rightarrow 0$  be a projective presentation of  $M$  and set

$$\vartheta_{\lambda}(M) := \text{Coker}((- , A^n) \rightarrow (- , A^m)).$$

For  $M' \in \text{mod-}A$  with projective presentation  $A^{n'} \xrightarrow{d'} A^{m'} \xrightarrow{\varepsilon'} M' \rightarrow 0$  and a morphism  $f : M \rightarrow M'$ , we have the commutative diagram

$$\begin{array}{ccccccc} A^n & \xrightarrow{d} & A^m & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\ \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\ A^{n'} & \xrightarrow{d'} & A^{m'} & \xrightarrow{\varepsilon'} & M' & \longrightarrow & 0. \end{array}$$

Then, Yoneda lemma in view of the fact that  $\mathcal{X}$  contains projectives, induces a natural transformation  $\vartheta_\lambda(f) : \vartheta_\lambda(M) \rightarrow \vartheta_\lambda(M')$  by the following commutative diagram

$$\begin{array}{ccccccc} (-, A^n) & \xrightarrow{(-, d)} & (-, A^m) & \longrightarrow & \vartheta_\lambda(M) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \\ (-, A^{n'}) & \xrightarrow{(-, d')} & (-, A^{m'}) & \longrightarrow & \vartheta_\lambda(M') & \longrightarrow & 0. \end{array}$$

A standard argument applies to show that  $\vartheta_\lambda(M)$  and  $\vartheta_\lambda(f)$  are independent of the choice of projective presentations of  $M$  and  $M'$  and also lifting of  $f$ .

Moreover, define

$$\vartheta_\rho(M) := (\text{mod-}A)(-, M)|_{\mathcal{X}} = \text{Hom}_A(-, M)|_{\mathcal{X}}.$$

We sometimes write  $(-, M)|_{\mathcal{X}}$  for  $\text{Hom}_A(-, M)|_{\mathcal{X}}$ , where it is clear from the context. Note that if  $M \in \mathcal{X}$ ,  $\text{Hom}_A(-, M)|_{\mathcal{X}} = \mathcal{X}(-, M)$ .

**Lemma 3.5.** *With the above assumptions, the functor  $\vartheta_\rho$  is full and faithful.*

*Proof.* Its faithfulness is easy and follows from the fact that  $\mathcal{X}$  contains projectives. We provide a proof for the fullness. Let  $\eta : \text{Hom}_A(-, A)|_{\mathcal{X}} \rightarrow \text{Hom}_A(-, A')|_{\mathcal{X}}$  be a morphism in  $\text{mod-}\mathcal{X}$ . Since  $\mathcal{X}$  is contravariantly finite, we have exact sequences  $X_1 \rightarrow X_0 \rightarrow A \rightarrow 0$  and  $X'_1 \rightarrow X'_0 \rightarrow A' \rightarrow 0$  of  $A$ -modules such that  $X_0, X'_0, X_1, X'_1 \in \mathcal{X}$  and the induced sequences

$$\begin{array}{ccccccc} \text{Hom}_A(-, X_1) & \longrightarrow & \text{Hom}_A(-, X_0) & \longrightarrow & \text{Hom}_A(-, A)|_{\mathcal{X}} & \longrightarrow & 0 \\ & & & & \downarrow \eta & & \\ \text{Hom}_A(-, X'_1) & \longrightarrow & \text{Hom}_A(-, X'_0) & \longrightarrow & \text{Hom}_A(-, A')|_{\mathcal{X}} & \longrightarrow & 0, \end{array}$$

are exact on  $\mathcal{X}$ . Since  $(-, X_i)$  is projectives for  $i = 0, 1$ ,  $\eta$  lifts to morphisms  $\eta_0 : \text{Hom}_A(-, X_0) \rightarrow \text{Hom}_A(-, X'_0)$  and  $\eta_1 : \text{Hom}_A(-, X_1) \rightarrow \text{Hom}_A(-, X'_1)$  making the diagram commutative. By Yoneda lemma, we get the commutative diagram

$$\begin{array}{ccccccc} X_1 & \longrightarrow & X_0 & \longrightarrow & A & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \\ X'_1 & \longrightarrow & X'_0 & \longrightarrow & A' & \longrightarrow & 0, \end{array}$$

This induces a morphism  $f : A \rightarrow A'$ . It is easy to check that  $\vartheta_\rho(f) = \eta$  and hence  $\vartheta_\rho$  is full.  $\square$

**Proposition 3.6.** *Let  $A$  be a right coherent ring and  $\mathcal{X}$  be a contravariantly finite subcategory of  $\text{mod-}A$  containing  $\text{prj-}A$ . Then the functors  $\vartheta_\lambda$  and  $\vartheta_\rho$  are respectively the left and the right adjoints of the functor  $\vartheta$  defined in Remark 3.2.*

*Proof.* Fix projective presentations  $(-, X_1) \xrightarrow{(-, d)} (-, X_0) \xrightarrow{\varepsilon} F \rightarrow 0$  and  $A^n \rightarrow A^m \rightarrow M \rightarrow 0$  of  $F \in \text{mod-}\mathcal{X}$  and  $M \in \text{mod-}A$ . We first show that  $\vartheta_\lambda$  is the left adjoint of  $\vartheta$ . Define

$$\varphi_{M,F} : \text{Hom}_A(M, e(F)) \rightarrow \text{Hom}_{\text{mod-}\mathcal{X}}(\vartheta_\lambda(M), F)$$

as follows. An  $A$ -morphism  $f : M \rightarrow \vartheta(F)$  can be lifted to commute the following diagram

$$\begin{array}{ccccccc} A^n & \longrightarrow & A^m & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\ X_1 & \xrightarrow{d} & X_0 & \xrightarrow{\pi} & \vartheta(F) & \longrightarrow & 0. \end{array}$$

Yoneda lemma helps us to have the following diagram in  $\text{mod-}\mathcal{X}$  such that the left square is commutative.

$$\begin{array}{ccccccc} (-, A^n) & \longrightarrow & (-, A^m) & \longrightarrow & \vartheta_\lambda(M) & \longrightarrow & 0 \\ \downarrow (-, f_1) & & \downarrow (-, f_0) & & & & \\ (-, X_1) & \longrightarrow & (-, X_0) & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

So it induces a map  $\sigma : \vartheta_\lambda(M) \rightarrow F$ . Define  $\varphi_{M,F}(f) := \sigma$ . Standard homological arguments guarantee that  $\varphi$  is well-defined. We show that it is an isomorphism. Assume that  $\varphi_{M,F}(f) = \sigma = 0$ . So there exists an  $A$ -morphism  $S = (-, s) : (-, A^m) \rightarrow (-, X_1)$  such that the lower triangle is commutative

$$\begin{array}{ccc} (-, A^n) & \longrightarrow & (-, A^m) \\ \downarrow (-, f_1) & \swarrow (-, s) & \downarrow (-, f_0) \\ (-, X_1) & \longrightarrow & (-, X_0) \end{array}$$

So by Yoneda we get the following diagram

$$\begin{array}{ccccccc} A^n & \longrightarrow & A^m & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow f_1 & \swarrow s & \downarrow f_0 & & \downarrow f & & \\ X_1 & \longrightarrow & X_0 & \longrightarrow & \vartheta(F) & \longrightarrow & 0. \end{array}$$

where the lower triangle is commutative. This in turn implies that  $f = 0$ . So  $\varphi_{M,F}$  is one to one. One can follow similar argument to see that  $\varphi_{M,F}$  is also surjective and hence is an isomorphism.

To show that  $\vartheta_\rho$  is the right adjoint of  $\vartheta$ , define with the same  $F$  and  $M$  as above,

$$\psi_{M,F} : \text{Hom}_A(\vartheta(F), M) \rightarrow \text{Hom}_{\text{mod-}\mathcal{X}}(F, \vartheta_\rho(M)),$$

by sending a morphism  $f : \vartheta(F) \rightarrow M$  to  $\vartheta(f)\delta$ , where  $\delta$  is the unique morphism that is obtained from the universal property of the cokernels in the following diagram

$$\begin{array}{ccccc} (-, X_1) & \xrightarrow{(-, d)} & (-, X_0) & \xrightarrow{\varepsilon} & F \longrightarrow 0 \\ & & \downarrow (-, \pi) & \swarrow \delta & \\ & & (-, \vartheta(F)) & & \end{array}$$



We claim that  $\psi_{M,F}$  is an isomorphism. Assume that  $\vartheta(f)\delta = 0$ . This in turn yields that  $(-, f)(-, \pi) = 0$ . So  $f\pi = 0$ , that implies  $f = 0$ , because  $\pi$  is a surjective map. Therefore  $\psi_{M,F}$  is a monomorphism.

To show that it is also surjective, pick a natural transformation  $\gamma : F \rightarrow \vartheta_\rho(M)$ . By Lemma 3.5,  $\gamma\varepsilon$  can be presented by a unique map say,  $g : X_0 \rightarrow M$ . But  $gd = 0$  and hence there exists a unique morphism  $h : \vartheta(F) \rightarrow M$  such that  $h\pi = g$ . It is obvious then that  $\psi_{M,F}(h) = \gamma$ .  $\square$

Set  $\text{mod}_0\text{-}\mathcal{X} := \text{Ker}\vartheta$ , the full subcategory of  $\text{mod-}\mathcal{X}$  consisting of all functors  $F$  such that  $\vartheta(F) = 0$ . This is equivalent to say that  $\text{mod}_0\text{-}\mathcal{X}$  consists of all functors that vanish on  $\Lambda$  or equivalently on all finitely generated projective  $\Lambda$ -modules. Since  $\vartheta$  is exact,  $\text{mod}_0\text{-}\mathcal{X}$  is a Serre subcategory of  $\text{mod-}\mathcal{X}$ . Moreover the inclusion functor  $\text{mod}_0\text{-}\mathcal{X} \rightarrow \text{mod-}\mathcal{X}$  is exact.

**Theorem 3.7.** *Let  $A$  be a right coherent ring and  $\mathcal{X}$  be a contravariantly finite subcategory of  $\text{mod-}A$  containing  $\text{prj-}A$ . Then there exists a recollement*

$$\begin{array}{ccccc} & \xleftarrow{i_\lambda} & & \xleftarrow{\vartheta_\lambda} & \\ \text{mod}_0\text{-}\mathcal{X} & \xrightarrow{i} & \text{mod-}\mathcal{X} & \xrightarrow{\vartheta} & \text{mod-}A \\ & \xleftarrow{i_\rho} & & \xleftarrow{\vartheta_\rho} & \end{array}$$

of abelian categories. In particular,

$$\frac{\text{mod-}\mathcal{X}}{\text{mod}_0\text{-}\mathcal{X}} \simeq \text{mod-}A.$$

*Proof.* By Proposition 3.6,  $\vartheta : \text{mod-}\mathcal{X} \rightarrow \text{mod-}A$  has a left and a right adjoint. Hence by Remark 2.1 to deduce that the recollement exists, we just need to show that either  $\vartheta_\lambda$  or  $\vartheta_\rho$ , and hence both of them, is full and faithful. This follows from Lemma 3.5. Hence the proof of the existence of recollement is complete. The equivalence is just an immediate consequence of the recollement. Hence we are done.  $\square$

The equivalence  $\frac{\text{mod-}\mathcal{X}}{\text{mod}_0\text{-}\mathcal{X}} \simeq \text{mod-}R$  will be called  $\mathcal{X}$ -Auslander's formula. In case  $\mathcal{X} = \text{mod-}R$ , we get the known (absolute) formula. Later on we will choose some specific classes and discuss some interesting examples.

Analogously Theorem 3.7 can be stated for  $\mathcal{X}\text{-mod}$ , the category of finitely presented covariant functors.

**Theorem 3.8.** *Let  $A$  be a right coherent ring and  $\mathcal{X}$  be a covariantly finite subcategory of  $\text{mod-}A$  containing  $\text{inj-}A$ . Then, there exists a recollement*

$$\begin{array}{ccccc} & \xleftarrow{i'_\lambda} & & \xleftarrow{\vartheta'_\lambda} & \\ \mathcal{X}\text{-mod}_0 & \xrightarrow{i'} & \mathcal{X}\text{-mod} & \xrightarrow{\vartheta'} & (\text{mod-}A)^{\text{op}} \\ & \xleftarrow{i'_\rho} & & \xleftarrow{\vartheta'_\rho} & \end{array}$$

of abelian categories, where  $\mathcal{X}\text{-mod}_0 = \text{Ker}\vartheta'$  is the full subcategory of  $\mathcal{X}\text{-mod}$  consisting of all functors that vanish on injective modules.

*Proof.* Let  $F \in \mathcal{X}\text{-mod}$ . Pick a projective presentation  $(X_1, -) \rightarrow (X_0, -) \rightarrow F \rightarrow 0$  of  $F$  and define  $\vartheta'(F) := \text{Ker}(X_0 \rightarrow X_1)$ .

On the other hand, for  $M \in \text{mod-}A$  define  $\vartheta'_\rho(M) := (M, -)|_{\mathcal{X}}$  and  $\vartheta'_\lambda(M) := \text{Coker}((I^1, -) \rightarrow (I^0, -))$ , where  $0 \rightarrow M \rightarrow I^0 \rightarrow I^1$  is an injective copresentation of  $M$ . One should now follow

similar, or rather dual, argument as we did, to prove that  $\vartheta'$  is exact,  $(\vartheta'_\lambda, \vartheta')$  and  $(\vartheta', \vartheta'_\rho)$  are adjoint pairs and  $\vartheta'_\rho$  is fully faithful and hence deduce from Remark 2.1 that recollement exists. We leave the details to the reader.  $\square$

**Remark 3.9.** By Remark 2.2, two exact sequences can be derived from a recollement of abelian categories. Here we explicitly study these two exact sequences attached to the recollement of Theorem 3.7.

Let  $F \in \text{mod-}\mathcal{X}$ . By the same notation as in the Remark 2.2 for the units and counits of adjunctions, we have the following two exact sequences

$$0 \longrightarrow ii_\rho(F) \xrightarrow{\eta_F^{ii_\rho}} F \xrightarrow{\delta_F^{\vartheta_\rho \vartheta}} \vartheta_\rho \vartheta(F) \longrightarrow \text{Coker}(\delta_F^{\vartheta_\rho \vartheta}) \longrightarrow 0;$$

$$0 \longrightarrow \text{Ker}(\eta_F^{\vartheta_\lambda \vartheta}) \longrightarrow \vartheta_\lambda \vartheta(F) \xrightarrow{\eta_F^{\vartheta_\lambda \vartheta}} F \xrightarrow{\delta_F^{ii_\lambda}} ii_\lambda(F) \longrightarrow 0.$$

As it is shown in the proof of Proposition 3.6,

$$\vartheta_\rho \vartheta(F) = (-, \vartheta(F))|_{\mathcal{X}}$$

and

$$\vartheta_\lambda \vartheta(F) = \text{Coker}((-, A^n) \longrightarrow (-, A^m)),$$

where  $A^n \longrightarrow A^m \longrightarrow \vartheta(F) \longrightarrow 0$  is a projective presentation of  $\vartheta(F)$ .

Clearly  $ii_\rho(F)$  and  $ii_\lambda(F)$  both are in  $\text{mod}_0\text{-}\mathcal{X}$ . Moreover, we may deduce from Remark 2.2 that  $\text{Coker}\delta_F^{\vartheta_\rho \vartheta}$  and  $\text{Ker}(\eta_F^{\vartheta_\lambda \vartheta})$  belong to  $\text{mod}_0\text{-}\mathcal{X}$ . Putting together, we obtain the exact sequences

$$0 \longrightarrow F_0 \longrightarrow F \longrightarrow (-, \vartheta(F))|_{\mathcal{X}} \longrightarrow F_1 \longrightarrow 0;$$

$$0 \longrightarrow F_2 \longrightarrow \vartheta_\lambda \vartheta(F) \longrightarrow F \longrightarrow F_3 \longrightarrow 0,$$

where  $F_0, F_1, F_2$  and  $F_3$  are in  $\text{mod}_0\text{-}\mathcal{X}$ .

It is worth to note that in case  $\mathcal{X} = \text{mod-}\Lambda$ , where  $\Lambda$  is an artin algebra, the first exact sequence is exactly the fundamental exact sequence obtained by Auslander [Au1, Page 203].

**Remark 3.10.** Let  $\mathcal{X}_1 \subseteq \mathcal{X}_2$  be contravariantly finite subcategories of  $\text{mod-}A$  that both contain projectives. Let  $F \in \text{mod-}\mathcal{X}_1$  and consider a projective presentation of  $F$

$$(-, X_1) \xrightarrow{(-, d)} (-, X_0) \longrightarrow F \longrightarrow 0.$$

Clearly we may write this presentation as

$$\text{Hom}_A(-, X_1)|_{\mathcal{X}_1} \longrightarrow \text{Hom}_A(-, X_0)|_{\mathcal{X}_1} \longrightarrow F \longrightarrow 0.$$

This allow us to extend  $F$  to  $\mathcal{X}_2$  and consider it as an object of  $\text{mod-}\mathcal{X}_2$  by setting

$$\tilde{F} = \text{Coker}(\text{Hom}_A(-, X_1)|_{\mathcal{X}_2} \longrightarrow \text{Hom}_A(-, X_0)|_{\mathcal{X}_2}).$$

Hence we can define a functor  $\xi : \text{mod-}\mathcal{X}_1 \longrightarrow \text{mod-}\mathcal{X}_2$  by  $\xi(F) = \tilde{F}$ . It can be checked easily that  $\xi| : \text{mod}_0\text{-}\mathcal{X}_1 \longrightarrow \text{mod}_0\text{-}\mathcal{X}_2$  is a functor and hence we have the following morphism of recollements

$$\begin{array}{ccccc} \text{mod}_0\text{-}\mathcal{X}_1 & \xrightleftharpoons{\quad} & \text{mod-}\mathcal{X}_1 & \xrightleftharpoons{\quad} & \text{mod-}A \\ \downarrow \xi| & & \downarrow \xi & & \downarrow \bar{\xi} \\ \text{mod}_0\text{-}\mathcal{X}_2 & \xrightleftharpoons{\quad} & \text{mod-}\mathcal{X}_2 & \xrightleftharpoons{\quad} & \text{mod-}A. \end{array}$$

In some sense the upper recollement is a sub-recollement of the lower one. Therefore, we have a partial order on the recollements and Auslander in a sense has considered the maximum case  $\mathcal{X} = \text{mod-}\Lambda$ .

**Remark 3.11.** Let  $\Lambda$  be a self-injective artin algebra over a commutative artinian ring  $R$ . By combining Theorems 3.7 and 3.8, we plan to construct auto-equivalences of  $\text{mod-}\Lambda$ . To this end, let  $\mathcal{X}$  be a functorially finite subcategory of  $\text{mod-}\Lambda$  containing  $\text{inj-}\Lambda = \text{prj-}\Lambda$ . By [AS, Theorem 2.3],  $\mathcal{X}$  is a dualising  $R$ -variety. So we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{mod}_0\text{-}\mathcal{X} & \longrightarrow & \text{mod-}\mathcal{X} & \longrightarrow & \frac{\text{mod-}\mathcal{X}}{\text{mod}_0\text{-}\mathcal{X}} \longrightarrow 0 \\ & & \downarrow D| & & \downarrow D & & \downarrow \overline{D} \\ 0 & \longrightarrow & \mathcal{X}\text{-mod}_0 & \longrightarrow & \mathcal{X}\text{-mod} & \longrightarrow & \frac{\mathcal{X}\text{-mod}}{\mathcal{X}\text{-mod}_0} \longrightarrow 0, \end{array}$$

where  $D$  is the usual duality of  $R$ -varieties. Hence the composition

$$D^{\mathcal{X}} : \text{mod-}\Lambda \xrightarrow{\overline{\vartheta}^{-1}} \frac{\text{mod-}\mathcal{X}}{\text{mod}_0\text{-}\mathcal{X}} \xrightarrow{\overline{D}} \frac{\mathcal{X}\text{-mod}}{\mathcal{X}\text{-mod}_0} \xrightarrow{\overline{\vartheta}'} (\text{mod-}\Lambda)^{\text{op}} \xrightarrow{\text{op}} \text{mod-}\Lambda,$$

denoted by  $\mathcal{D}^{\mathcal{X}}$  is an auto-equivalence of  $\text{mod-}\Lambda$  with respect to  $\mathcal{X}$ . Note that if  $\mathcal{X} = \text{prj-}\Lambda$ , then  $D^{\text{prj-}\Lambda}$  is the identity functor.

#### 4. EXAMPLES AND APPLICATIONS

In this section, we provide some examples as well as applications of the recollements introduced in the previous section. Throughout the section  $\Lambda$  is an artin algebra. Let  $\mathcal{X}$  be a full subcategory of  $\text{mod-}\Lambda$ . The set of isoclasses of indecomposable modules of  $\mathcal{X}$  will be denoted by  $\text{Ind-}\mathcal{X}$ .  $\mathcal{X}$  is called of finite type if  $\text{Ind-}\mathcal{X}$  is a finite set.  $\Lambda$  is called of finite representation type if  $\text{mod-}\Lambda$  is of finite type. If  $\mathcal{X}$  is of finite type then it admits a representation generator, i.e. there exists  $X \in \mathcal{X}$  such that  $\mathcal{X} = \text{add-}X$ . It is known that  $\text{add-}X$  is a contravariantly finite subcategory of  $\text{mod-}\Lambda$ . Set  $\Gamma(\mathcal{X}) = \text{End}_{\Lambda}(X)$ . Clearly  $\Gamma(\mathcal{X})$  is an artin algebra. It is known that the evaluation functor  $\zeta_X : \text{mod-}\mathcal{X} \rightarrow \text{mod-}\Gamma(\mathcal{X})$  defined by  $\zeta_X(F) = F(X)$ , for  $F \in \text{mod-}\mathcal{X}$ , is an equivalence of categories. It also induces an equivalence of categories  $\text{mod-}\underline{\mathcal{X}}$  and  $\text{mod-}\Gamma(\underline{\mathcal{X}})$ . Recall that  $\Gamma(\underline{\mathcal{X}}) = \text{End}_{\Lambda}(X)/\mathcal{P}$ , where  $\mathcal{P}$  is the ideal of  $\Gamma(\mathcal{X})$  including morphisms factoring through projective modules.

The artin algebra  $\Gamma(\mathcal{X})$ , resp.  $\Gamma(\underline{\mathcal{X}})$ , is called relative, resp. stable, Auslander algebra of  $\Lambda$  with respect to the subcategory  $\mathcal{X}$ .

We need the following result in this section. Let  $\pi : \mathcal{X} \rightarrow \underline{\mathcal{X}}$  be the canonical functor. It then induces an exact functor  $\mathfrak{F} : \text{Mod-}\underline{\mathcal{X}} \rightarrow \text{Mod-}\mathcal{X}$ . It is not hard to see that  $\mathfrak{F}$  in turn induces an equivalence between  $\text{Mod-}\underline{\mathcal{X}}$  and the full subcategory of  $\text{Mod-}\mathcal{X}$  consisting of functors vanishing on  $\text{Prj-}\mathcal{A}$ .

**Proposition 4.1.** *Let  $\mathcal{A}$  be an abelian category with enough projective objects and  $\mathcal{X}$  be a subcategory of  $\mathcal{A}$  containing  $\text{Prj-}\mathcal{A}$ . Then we have the following commutative diagram*

$$\begin{array}{ccc} \text{Mod-}\underline{\mathcal{X}} & \xrightarrow{\mathfrak{F}} & \text{Mod-}\mathcal{X} \\ \uparrow & & \uparrow \\ \text{mod-}\underline{\mathcal{X}} & \xrightarrow{\mathfrak{F}|} & \text{mod}_0\text{-}\mathcal{X}, \end{array}$$

such that the lower row is an equivalence and the others are inclusions. If furthermore  $\mathcal{X}$  is contravariantly finite, then  $\text{mod-}\underline{\mathcal{X}}$  is an abelian category with enough projective objects.

*Proof.* Pick  $F \in \text{mod-}\underline{\mathcal{X}}$ . Clearly  $\mathfrak{F}(F)$  vanishes on projective objects, so to show that  $\mathfrak{F}(F) \in \text{mod}_0\text{-}\mathcal{X}$ , we just need to show that  $\mathfrak{F}(F) \in \text{mod-}\mathcal{X}$ . To this end, since  $\mathfrak{F}$  is an exact functor, it is enough to show that  $\mathfrak{F}(\underline{\mathcal{X}}(-, \underline{X})) \in \text{mod-}\mathcal{X}$ , for any  $X \in \mathcal{X}$ . We let  $(-, \underline{X})$  denote the image of  $\underline{\mathcal{X}}(-, \underline{X})$  under  $\mathfrak{F}$ . Since  $\mathcal{A}$  has enough projective objects, for  $X \in \mathcal{X}$ , there exists a short exact sequence

$$0 \rightarrow \Omega(X) \rightarrow P \rightarrow X \rightarrow 0,$$

where  $P$  is a projective object. Then, we get the following exact sequence

$$0 \rightarrow (-, \Omega(X))|_{\mathcal{X}} \rightarrow (-, P) \rightarrow (-, X) \rightarrow (-, \underline{X}) \rightarrow 0$$

in  $\text{Mod-}\mathcal{X}$ . Hence,  $(-, \underline{X}) \in \text{mod-}\mathcal{X}$ , since  $\mathcal{X}$  contains  $\text{Prj-}\mathcal{A}$ . On the other hand, let  $F \in \text{mod}_0\text{-}\mathcal{X}$  and take a projective presentation  $(-, X_1) \xrightarrow{(-, d)} (-, X_0) \rightarrow F \rightarrow 0$  of  $F$ . Since  $F \in \text{mod}_0\text{-}\mathcal{X}$ ,  $d : X_1 \rightarrow X_0$  is surjective and hence we have the following commutative diagram

$$\begin{array}{ccccccc} (-, X_1) & \xrightarrow{(-, d)} & (-, X_0) & \longrightarrow & F & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ (-, \underline{X}_1) & \xrightarrow{(-, \underline{d})} & (-, \underline{X}_0) & \longrightarrow & F & \longrightarrow & 0. \end{array}$$

Note that the two vertical natural transformations on the left attach to any morphism  $X \rightarrow X_1$  and  $X \rightarrow X_0$ , its residue class modulo morphisms factoring through projective objects. As  $\mathfrak{F}$  is an equivalence of categories, there exists morphism  $d'$  such that  $\mathfrak{F}(d') = (-, \underline{d})$ . Set

$$F' := \text{Coker}(\underline{\mathcal{X}}(-, \underline{X}_1) \xrightarrow{d'} \underline{\mathcal{X}}(-, \underline{X}_0))$$

Then  $\mathfrak{F}(F') = F$ , since  $\mathfrak{F}$  is an exact functor. The proof of this part is hence complete. It is now plain that  $\text{mod-}\underline{\mathcal{X}}$  is an abelian category.  $\square$

Note that in [MT] special subcategories  $\mathcal{X}$  of an abelian category  $\mathcal{A}$ , called quasi-resolving subcategories, have been studied with the property that  $\text{mod-}\underline{\mathcal{X}}$  is still an abelian category.  $\mathcal{X}$  is called a quasi-resolving subcategory if it contains the projective objects and closed under finite direct sums and kernels of epimorphisms.

**4.1. Some examples.** Now we are ready to investigate some examples.

**Example 4.1.1.** Let  $\mathcal{X} = \text{add-}X$  be a subcategory of  $\text{mod-}\Lambda$  such that  $\Lambda$  is a summand of  $X$ . Hence  $\mathcal{X}$  is a contravariantly finite subcategory of  $\text{mod-}\Lambda$  containing  $\text{prj-}\Lambda$ . So Theorem 3.7 applies to show, in view of Proposition 4.1, that the following recollement exists.

$$\begin{array}{ccccc} & \overset{\zeta_{\underline{\mathcal{X}}} i_{\lambda} \zeta_{\underline{\mathcal{X}}}^{-1}}{\curvearrowright} & & \overset{\zeta_X \vartheta_{\lambda}}{\curvearrowright} & \\ \text{mod-}\Gamma(\underline{\mathcal{X}}) & \xrightarrow{\zeta_{\underline{\mathcal{X}}}^{-1} i_{\zeta_X}} & \text{mod-}\Gamma(\mathcal{X}) & \xrightarrow{\vartheta \zeta_X^{-1}} & \text{mod-}\Lambda \\ & \underset{\zeta_{\underline{\mathcal{X}}} i_{\rho} \zeta_{\underline{\mathcal{X}}}^{-1}}{\curvearrowright} & & \underset{\zeta_X \vartheta_{\rho}}{\curvearrowright} & \end{array}$$

It is routine to check that this recollement is equivalent to the one presented in [Ps, Example 2.10]. So in this case, we just give a functor category approach to the existence of this recollement.

**Example 4.1.2.** As a particular case of the above example, let  $\Lambda$  be of finite representation type. Then, in view of Proposition 4.1, we have the recollement

$$\begin{array}{ccccc} & \xleftarrow{i_\lambda} & & \xleftarrow{\vartheta_\lambda} & \\ \text{mod-}\Gamma(\underline{\text{mod-}\Lambda}) & \xrightarrow{i} & \text{mod-}\Gamma(\text{mod-}\Lambda) & \xrightarrow{\vartheta} & \text{mod-}\Lambda \\ & \xleftarrow{i_\rho} & & \xleftarrow{\vartheta_\rho} & \end{array}$$

It is interesting to note that in this recollement Auslander algebra, stable Auslander algebra and the algebra itself are appeared.

**Example 4.1.3.** Recall that a  $\Lambda$ -module  $G$  is called Gorenstein projective if it is a syzygy of a  $\text{Hom}_\Lambda(-, \text{Prj-}\Lambda)$ -exact exact complex

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots,$$

of projective modules. The class of Gorenstein projective modules is denoted by  $\text{GPrj-}\Lambda$ . Dually one can define the class of Gorenstein injective modules  $\text{GInj-}\Lambda$ . We set  $\text{Gprj-}\Lambda = \text{GPrj-}\Lambda \cap \text{mod-}\Lambda$  and  $\text{Ginj-}\Lambda = \text{GInj-}\Lambda \cap \text{mod-}\Lambda$ .  $\Lambda$  is called virtually Gorenstein if  $(\text{GPrj-}\Lambda)^\perp = {}^\perp(\text{GInj-}\Lambda)$ , where orthogonal is taken with respect to  $\text{Ext}^1$ , see [BR]. It is proved by Beligiannis [Be, Proposition 4.7] that if  $\Lambda$  is a virtually Gorenstein algebra, then  $\text{Gprj-}\Lambda$  is a contravariantly finite subcategory of  $\text{mod-}\Lambda$ .

Let  $\Lambda$  be a virtually Gorenstein algebra and set  $\mathcal{X} = \text{Gprj-}\Lambda$ . Hence  $\text{Gprj-}\Lambda$  is contravariantly finite and obviously contains  $\text{prj-}\Lambda$ . So Theorem 3.7 applies and again in view of Proposition 4.1, we get the following recollement

$$\begin{array}{ccccc} & \xleftarrow{i_\lambda} & & \xleftarrow{\vartheta_\lambda} & \\ \text{mod-}\Gamma(\underline{\text{Gprj-}\Lambda}) & \xrightarrow{i} & \text{mod-}\Gamma(\text{Gprj-}\Lambda) & \xrightarrow{\vartheta} & \text{mod-}\Lambda \\ & \xleftarrow{i_\rho} & & \xleftarrow{\vartheta_\rho} & \end{array}$$

Recall that an algebra  $\Lambda$  is called of finite Cohen-Macaulay type, CM-finite for short, if  $\text{Gprj-}\Lambda$  is of finite type. Assume that  $\Lambda$  is a CM-finite Gorenstein algebra. Then by [E, Corollary 3.5],  $\Gamma(\underline{\text{Gprj-}\Lambda})$  is a self-injective algebra and by [Be, Corollary 6.8(v)]  $\Gamma(\text{Gprj-}\Lambda)$  is of finite global dimension. Hence, in this case, we have a recollement including three types of algebras: self-injective, finite global dimension and Gorenstein.

**Example 4.1.4.** Let  $n \geq 1$ . Roughly speaking a subcategory  $\mathcal{X}$  of  $\text{mod-}\Lambda$  is called  $n$ -cluster tilting if it is functorially finite and the pair  $(\mathcal{X}, \mathcal{X})$  forms a cotorsion pair with respect to  $\text{Ext}^i$  for  $0 < i < n$ , see [I2, Definition 1.1] for the exact definition. Obviously, an  $n$ -cluster tilting subcategory  $\mathcal{X}$  of  $\text{mod-}\Lambda$  satisfies the conditions of Theorem 3.7 and so we have a recollement with respect to  $\mathcal{X}$ . Note that this fact also has been announced by Yasuaki Ogawa in ICRA 2016, see the abstract book of ICRA 17th, page 34.

**Remark 4.1.5.** Above examples show that for many different subcategories  $\mathcal{X}$  of  $\text{mod-}\Lambda$ , we have a relative Auslander's formula, i.e an equivalence  $\frac{\text{mod-}\mathcal{X}}{\text{mod}_0\text{-}\mathcal{X}} \simeq \text{mod-}\Lambda$ . At least with some extra assumptions on the algebra, we may guarantee that the functor category  $\text{mod-}\mathcal{X}$  has similar nice homological properties as in Auslander's cases, i.e.  $\mathcal{X} = \text{mod-}(\text{mod-}\Lambda)$ . For example, if we assume that  $\Lambda$  is a 1-Gorenstein algebra, then  $\text{Gprj-}\Lambda$  is closed under submodules and hence  $\text{mod-}(\text{Gprj-}\Lambda)$  has global dimension at most 2, like Auslander's result. Therefore, it seems worth to study this case more explicitly.

**4.2. Applications.** Study of Morita equivalence of two algebras through the study of Morita equivalence of related algebras has some precedents in the literature, see e.g. [HT] and [KY]. As applications of the recollement of Theorem 3.7, we present two results in this direction. We precede them with a lemma, that is of independent interest, as it provides a description for functors in  $\text{mod}_0\text{-}\mathcal{X}$ .

**Lemma 4.2.1.** *Let  $\mathcal{X}$  be a contravariantly finite subcategory of  $\text{mod-}A$  containing all finitely generated projective modules, where as usual  $A$  is a right coherent ring. Let  $F \in \text{mod-}\mathcal{X}$ . Then  $F \in \text{mod}_0\text{-}\mathcal{X}$  if and only if  $(F, (-, M)|_{\mathcal{X}}) = 0$ , for all  $M \in \text{mod-}\Lambda$ . Moreover  $\text{Ext}^1(F, (-, M)|_{\mathcal{X}}) = 0$ , for all  $F \in \text{mod}_0\text{-}\mathcal{X}$  and all  $M \in \text{mod-}\Lambda$ .*

*Proof.* Let  $F \in \text{mod}_0\text{-}\mathcal{X}$  and pick  $M \in \text{mod-}\Lambda$ . Consider epimorphism  $(-, X) \xrightarrow{\varepsilon} F \rightarrow 0$ . Let  $\eta \in (F, (-, M)|_{\mathcal{X}})$ . Since  $F(\Lambda) = 0$ ,  $\eta_{\Lambda}\varepsilon_{\Lambda} = 0$ . So clearly  $\eta\varepsilon = 0$ . This, in turn, implies that  $\eta = 0$ , because  $\varepsilon$  is an epimorphism.

Conversely, assume that  $(F, (-, M)|_{\mathcal{X}}) = 0$ , for all  $M \in \text{mod-}\Lambda$ . By Remark 3.9, we have the exact sequence

$$0 \longrightarrow F_0 \longrightarrow F \xrightarrow{\vartheta_F} (-, \vartheta(F))|_{\mathcal{X}} \longrightarrow F_1 \longrightarrow 0$$

such that  $F_0$  and  $F_1$  are in  $\text{mod}_0\text{-}\mathcal{X}$ . By assumption  $\vartheta_F = 0$ . Therefore  $F = F_0$ . The proof is hence complete.

Now assume that  $F \in \text{mod}_0\text{-}\mathcal{X}$ . We show that  $\text{Ext}^1(F, (-, M)|_{\mathcal{X}}) = 0$  for all  $M \in \text{mod-}\Lambda$ . Consider a projective presentation

$$(-, X_1) \xrightarrow{(-, d)} (-, X_0) \xrightarrow{\varepsilon} F \longrightarrow 0$$

of  $F$ , with  $X_1$  and  $X_0$  in  $\mathcal{X}$ . Set  $K = \text{Ker}d$ . So we get the following exact sequence

$$0 \longrightarrow (-, K)|_{\mathcal{X}} \longrightarrow (-, X_1) \longrightarrow (-, X_0) \xrightarrow{\varepsilon} F \longrightarrow 0$$

with  $(-, X_1)$  and  $(-, X_0)$  projectives. Hence, for  $M \in \text{Mod-}\Lambda$ ,  $\text{Ext}^1(F, (-, M)|_{\mathcal{X}})$  can be calculated by the deleted sequence

$$0 \longrightarrow (-, K)|_{\mathcal{X}} \longrightarrow (-, X_1) \longrightarrow (-, X_0) \longrightarrow 0.$$

Pick  $M \in \text{mod-}A$  and apply the functor  $(-, (-, M)|_{\mathcal{X}})$  on this sequence to obtain the following sequence

$$0 \longrightarrow ((-, X_0), (-, M)|_{\mathcal{X}}) \longrightarrow ((-, X_1), (-, M)|_{\mathcal{X}}) \longrightarrow ((-, K)|_{\mathcal{X}}, (-, M)|_{\mathcal{X}}).$$

So, to complete the proof, it is enough to show that this sequence is exact. This we do. Since by Lemma 3.5,  $\vartheta_{\rho}$  is full and faithful, we deduce that the vertical maps of the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_{\Lambda}(X_0, M) & \longrightarrow & \text{Hom}_{\Lambda}(X_1, M) & \longrightarrow & \text{Hom}_{\Lambda}(K, M) \\ & & \downarrow \vartheta_{\rho} & & \downarrow \vartheta_{\rho} & & \downarrow \vartheta_{\rho} \\ 0 & \longrightarrow & ((-, X_0), (-, M)|_{\mathcal{X}}) & \longrightarrow & ((-, X_1), (-, M)|_{\mathcal{X}}) & \longrightarrow & ((-, K)|_{\mathcal{X}}, (-, M)|_{\mathcal{X}}) \end{array}$$

are isomorphisms. On the other hand, since  $F \in \text{mod}_0\text{-}\mathcal{X}$ , the sequence  $0 \longrightarrow K \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0$  is exact. So the left exactness of  $\text{Hom}_{\Lambda}(-, M)$ , implies the exactness of the upper row. Hence the lower row is exact and we get the result.  $\square$

**Proposition 4.2.2.** *Let  $\Lambda$ , resp.  $\Lambda'$ , be artin algebras. Let  $\mathcal{X} \subseteq \text{mod-}\Lambda$ , resp.  $\mathcal{X}' \subseteq \text{mod-}\Lambda'$ , be subcategories of finite type such that  $\Lambda, D(\Lambda) \in \mathcal{X}$ , resp.  $\Lambda', D(\Lambda') \in \mathcal{X}'$ . Then  $\Lambda$  and  $\Lambda'$  are*

*Morita equivalent if  $\Gamma(\mathcal{X})$  and  $\Gamma(\mathcal{X}')$  are Morita equivalent. In particular, in this situation  $\Gamma(\underline{\mathcal{X}})$  and  $\Gamma(\underline{\mathcal{X}'})$  are also Morita equivalent.*

*Proof.* Since,  $\mathcal{X}$  and  $\mathcal{X}'$  are of finite type, they are contravariantly finite subcategories of  $\text{mod-}\Lambda$  and  $\text{mod-}\Lambda'$ , respectively. Moreover, they both are containing projectives. So Theorem 3.7, applies. Assume that  $\Gamma(\mathcal{X})$  and  $\Gamma(\mathcal{X}')$  are Morita equivalent and  $\Phi : \Gamma(\mathcal{X}) \rightarrow \Gamma(\mathcal{X}')$  denote the equivalence. Let  $\Phi : \text{mod-}\Gamma(\mathcal{X}) \rightarrow \text{mod-}\Gamma(\mathcal{X}')$  denote the equivalence which of course is an exact functor. In view of the related recollements of  $\mathcal{X}$  and  $\mathcal{X}'$  obtained from Theorem 3.7, to get the proof, it suffices to prove that  $\Phi(F) \in \text{mod}_0\text{-}\mathcal{X}'$ , for each  $F \in \text{mod}_0\text{-}\mathcal{X}$  and similarly for its quasi-inverse  $\Psi = \Phi^{-1}$ . By symmetry we just prove it for  $\Phi$ . Let  $F \in \text{mod}_0\text{-}\mathcal{X}$ . By the above lemma, to show that  $\Phi(F) \in \text{mod}_0\text{-}\mathcal{X}'$ , we show that  $(\Phi(F), (-, M')|_{\mathcal{X}'}) = 0$ , for all  $M' \in \text{mod-}\Lambda'$ . To do this, pick  $M \in \text{mod-}\Lambda'$ . Since  $D(\Lambda') \in \mathcal{X}'$ , there exists a monomorphism  $0 \rightarrow M' \rightarrow I'$  in  $\text{mod-}\Lambda'$  with  $I' \in \text{inj-}\Lambda'$ . So there is a monomorphism

$$0 \rightarrow (-, M')|_{\mathcal{X}'} \xrightarrow{\delta} (-, I')$$

in  $\text{mod-}\mathcal{X}'$ . Note that  $(-, I')$  is in fact a projective object in  $\text{mod-}\mathcal{X}'$ , because  $D(\Lambda') \in \mathcal{X}'$ . Since  $\Phi$  preserves projective functors we get monomorphism  $0 \rightarrow \Phi^{-1}((-, M)|_{\mathcal{X}}) \xrightarrow{\Phi^{-1}(\delta)} (-, X)$  in  $\text{mod-}\mathcal{X}$  with  $X \in \mathcal{X}$ . Now for any  $\eta \in (\Phi(F), (-, M)|_{\mathcal{X}'})$ ,  $\Phi^{-1}(\delta)\Phi^{-1}(\eta) \in (F, (-, X))$  should be zero, since  $F \in \text{mod}_0\text{-}\mathcal{X}$ , see Lemma 4.2.1. Hence  $\delta\eta = 0$ . This implies that  $\eta = 0$  since  $\delta$  is a monomorphism. It is now plain that  $\Gamma(\underline{\mathcal{X}})$  and  $\Gamma(\underline{\mathcal{X}'})$  are also Morita equivalent. The proof is hence complete.  $\square$

It has been proved by Auslander [Au2] that for an arbitrary artin algebra  $\Lambda$  there exists an artin algebra  $\tilde{\Lambda}$  of finite global dimension and an idempotent  $\xi$  of  $\tilde{\Lambda}$  such that  $\Lambda = \xi\tilde{\Lambda}\xi$ . Therefore, artin algebras of finite global dimension determine all artin algebras. To construct the algebra  $\tilde{\Lambda}$ , let  $J$  be the radical of  $\Lambda$  and  $n$  be its nilpotency index. Set  $M := \bigoplus_{1 \leq i \leq n} \frac{\Lambda}{J^i}$ , as right  $\Lambda$ -module. Then  $\tilde{\Lambda} = \text{End}_{\Lambda}(M)$ . We throughout call  $\tilde{\Lambda}$  the  $\Lambda$ -algebra of  $\Lambda$ , where ‘ $\Lambda$ ’ stands both for ‘Auslander’ and also ‘Associated’.

As a corollary of Proposition 4.2.2 we have the following result.

**Corollary 4.2.3.** *Let  $\Lambda$  and  $\Lambda'$  be self-injective artin algebras. If their  $\Lambda$ -algebras  $\tilde{\Lambda}$  and  $\tilde{\Lambda}'$  are Morita equivalent, then so are  $\Lambda$  and  $\Lambda'$ . In this case, stable  $\Lambda$ -algebras of  $\Lambda$  and  $\Lambda'$  are also Morita equivalent.*

*Proof.* Let  $\tilde{\Lambda} = \text{End}_{\Lambda}(M) = \Gamma(\text{add-}M)$  and  $\tilde{\Lambda}' = \text{End}_{\Lambda'}(M') = \Gamma(\text{add-}M')$ . Set  $\mathcal{X} = \text{add-}M$  and  $\mathcal{X}' = \text{add-}M'$ . So  $\text{mod-}\tilde{\Lambda} \simeq \text{mod-}\mathcal{X}$  and  $\text{mod-}\tilde{\Lambda}' \simeq \text{mod-}\mathcal{X}'$ . Since for self-injective algebras the subcategories of projective and injective modules coincide and  $\Lambda \in \mathcal{X}$ , resp.  $\Lambda' \in \mathcal{X}'$ , we deduce that  $D(\Lambda) \in \mathcal{X}$ , resp.  $D(\Lambda') \in \mathcal{X}'$ . Now the result follows immediately from the above proposition.  $\square$

**Remark 4.2.4.** Let  $F$  and  $F'$  be functors in  $\text{mod-}\mathcal{X}$ . By Remark 3.9 there exists exact sequences

$$0 \rightarrow F_0 \rightarrow F \rightarrow (-, \vartheta(F))|_{\mathcal{X}} \rightarrow F_1 \rightarrow 0;$$

$$0 \rightarrow F'_0 \rightarrow F' \rightarrow (-, \vartheta(F'))|_{\mathcal{X}} \rightarrow F'_1 \rightarrow 0;$$

such that  $F_0, F_1, F'_0$  and  $F'_1$  are in  $\text{mod}_0\text{-}\mathcal{X}$ . Note that Lemma 4.2.1 allow us to follow similar argument as in the Proposition 3.4 of [Aul] and deduce that

$$(F, (-, \vartheta(F'))|_{\mathcal{X}}) \cong ((-, \vartheta(F))|_{\mathcal{X}}, (-, \vartheta(F'))|_{\mathcal{X}}).$$

So for a morphism  $\sigma : F \rightarrow F'$  in  $\text{mod-}\mathcal{X}$ , there exists a unique map  $\delta : (-, \vartheta(F))|_{\mathcal{X}} \rightarrow (-, \vartheta(F'))|_{\mathcal{X}}$  commuting the square

$$\begin{array}{ccc} F & \longrightarrow & (-, \vartheta(F))|_{\mathcal{X}} \\ \downarrow \sigma & & \downarrow \delta \\ F' & \longrightarrow & (-, \vartheta(F'))|_{\mathcal{X}} \end{array}$$

Consequently, there are unique morphisms  $\sigma_0 : F_0 \rightarrow F'_0$  and  $\sigma_1 : F_1 \rightarrow F'_1$  such that the following diagram is commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_0 & \longrightarrow & F & \longrightarrow & (-, \vartheta(F))|_{\mathcal{X}} & \longrightarrow & F_1 & \longrightarrow & 0 \\ & & \downarrow \sigma_0 & & \downarrow \sigma & & \downarrow \delta & & \downarrow \sigma_1 & & \\ 0 & \longrightarrow & F'_0 & \longrightarrow & F' & \longrightarrow & (-, \vartheta(F'))|_{\mathcal{X}} & \longrightarrow & F'_1 & \longrightarrow & 0. \end{array}$$

It is not difficult to see that  $\delta = (-, \vartheta(\sigma))|_{\mathcal{X}}$ .

## 5. COVARIANT FUNCTORS

Throughout this section, assume that  $\mathcal{X}$  is a functorially finite subcategory of  $\text{mod-}\Lambda$  containing projectives, where as before  $\Lambda$  is an artin algebra over a commutative artinian ring  $R$ . The aim of this section is to construct analogously a recollement involving the category of finitely presented covariant functors  $\mathcal{X}\text{-mod}$  and the category of left  $\Lambda$ -modules. To this end, we use the structure of injective objects in  $\mathcal{X}\text{-mod}$  and follow the general argument as in the proof of Theorem 3.7, i.e. introduce three appropriate functors that are mutually adjoints and apply Remark 2.1. Since injectives of  $\mathcal{X}\text{-mod}$  play a significant role in the functors appearing in this recollement, we study them in a subsection with some details.

**Set up.** Throughout the section,  $\Lambda$  is an artin algebra and  $\mathcal{X}$  is a functorially finite subcategory of  $\text{mod-}\Lambda$  containing  $\text{prj-}\Lambda$ .

**5.1. Injective finitely presented covariant functors.** Let  $A$  be an arbitrary ring. Let  $(\text{mod-}A)\text{-mod}$  denote the subcategory of  $(\text{mod-}A)\text{-Mod}$  consisting of finitely presented covariant functors on  $\text{mod-}A$ .

**5.1.1.** It is proved by Auslander [Au1, Lemma 6.1] that for a left  $A$ -module  $M$ , the covariant functor  $- \otimes_A M$  is finitely presented if and only if  $M$  is a finitely presented left  $A$ -module. It is known that there is a full and faithful functor  $T : A\text{-mod} \rightarrow (\text{mod-}A)\text{-mod}$  defined by the attachment  $M \mapsto (- \otimes_A M)|_{\text{mod-}A}$ . Gruson and Jensen [GJ, 5.5] showed that the category  $(\text{mod-}A)\text{-mod}$  has enough injective objects and injectives are exactly those functors isomorphic to a functor of the form  $- \otimes_A M$ , for some left  $A$ -module  $M$ , see also [Pr, Proposition 2.27].

Our aim in this subsection is to study  $\text{inj-}(\mathcal{X}\text{-mod})$ .

**5.1.2.** We begin by considering the functor

$$t := t_{\mathcal{X}} : \Lambda\text{-mod} \rightarrow \mathcal{X}\text{-mod}$$



defined by the attachment  $M \mapsto (- \otimes_A M)|_{\mathcal{X}}$ . It is easy to see that  $(- \otimes_{\Lambda} M)|_{\mathcal{X}} \in \mathcal{X}\text{-mod}$ . The proof is similar to [Au1, Lemma 6.1]: one should apply the functorial isomorphism

$$- \otimes_{\Lambda} P \simeq (\text{Hom}_{\Lambda}(P, \Lambda), -),$$

where  $P \in \text{prj-}\Lambda$ , to the first two terms of the exact sequence

$$- \otimes_{\Lambda} \Lambda^n \rightarrow - \otimes_{\Lambda} \Lambda^m \rightarrow M \rightarrow 0,$$

that is induced from a projective presentation  $\Lambda^n \rightarrow \Lambda^m \rightarrow M \rightarrow 0$  of  $M$ . Obviously  $t$  sends any morphism  $f : M \rightarrow M'$  of left  $\Lambda$ -modules to  $(- \otimes f)|_{\mathcal{X}}$ .

**Lemma 5.1.3.** *The functor  $t$  defined above is full and faithful.*

*Proof.* By definition, it is plain that  $t$  is a faithful functor. To prove that it is full, consider a natural transformation  $\eta : (- \otimes M)|_{\mathcal{X}} \rightarrow (- \otimes M')|_{\mathcal{X}}$ . There exists a morphism  $h : M \rightarrow M'$  that commutes the following diagram

$$\begin{array}{ccc} M & \xrightarrow{h} & M' \\ \downarrow \wr & & \downarrow \wr \\ \Lambda \otimes_{\Lambda} M & \xrightarrow{\eta_{\Lambda}} & \Lambda \otimes_{\Lambda} M'. \end{array}$$

Therefore  $\eta_{\Lambda} \simeq \Lambda \otimes_{\Lambda} h$ . This equality can be extended easily to  $\Lambda^n$ , that is,  $\eta_{\Lambda^n} = \Lambda^n \otimes_{\Lambda} h$ . Now let  $X$  be an arbitrary right  $\Lambda$ -module. Consider a projective presentation  $\Lambda^n \rightarrow \Lambda^m \rightarrow X \rightarrow 0$  of  $X$ . It follows from the following diagram

$$\begin{array}{ccccccc} \Lambda^n \otimes_{\Lambda} M & \longrightarrow & \Lambda^m \otimes_{\Lambda} M & \longrightarrow & X \otimes_{\Lambda} M & \longrightarrow & 0 \\ \downarrow \Lambda^n \otimes_{\Lambda} h & & \downarrow \Lambda^m \otimes_{\Lambda} h & & \downarrow \eta_X & & \\ \Lambda^n \otimes_{\Lambda} M' & \longrightarrow & \Lambda^m \otimes_{\Lambda} M' & \longrightarrow & X \otimes_{\Lambda} M' & \longrightarrow & 0. \end{array}$$

that  $\eta_X \simeq X \otimes_{\Lambda} h$ . So  $t(h) = \eta$ . This completes the proof.  $\square$

**5.1.4.** By Remark 2.4,  $\text{mod-}\Lambda$  and also  $\mathcal{X}$  are dualising  $R$ -varieties. So duality  $D = \text{Hom}_R(-, E)$  induces the following commutative diagram

$$\begin{array}{ccc} \text{mod}(\text{mod-}\Lambda) & \xrightarrow{D_{\text{mod-}\Lambda}} & (\text{mod-}\Lambda)\text{-mod} \\ \downarrow |_{\mathcal{X}} & & \downarrow |_{\mathcal{X}} \\ \text{mod-}\mathcal{X} & \xrightarrow{D_{\mathcal{X}}} & \mathcal{X}\text{-mod} \end{array}$$

where the rows are duality and columns are restrictions.

Since  $D_{\text{mod-}\Lambda}$  is a duality, it sends projective objects to the injective ones. Hence, in view of 5.1.1, we may deduce that for  $M \in \text{mod-}\Lambda$ , there exists a left  $\Lambda$ -module  $M'$  such that  $D_{\text{mod-}\Lambda}(\text{Hom}_{\Lambda}(-, M)) \simeq - \otimes_{\Lambda} M'$ .  $M'$  is uniquely determined up to isomorphism, thanks to the faithfulness of the functor  $T$ .

This isomorphism can be restricted to  $\mathcal{X}$ , to induce the following isomorphism

$$D_{\text{mod-}\Lambda}(\text{Hom}_{\Lambda}(-, M))|_{\mathcal{X}} \simeq (- \otimes_{\Lambda} M')|_{\mathcal{X}}.$$

In case  $X \in \mathcal{X}$ , this can be written more simply as

$$D_{\mathcal{X}}((- , M)) \simeq (- \otimes_{\Lambda} M')|_{\mathcal{X}}.$$

Therefore, we have the following proposition.

**Proposition 5.1.5.** *Let  $\Lambda$  be an artin algebra and  $\mathcal{X}$  be a functorially finite subcategory of  $\text{mod-}\Lambda$  containing  $\text{prj-}\Lambda$ . Then  $\mathcal{X}\text{-mod}$  has enough injectives. Injective objects are those functors of the form  $(-\otimes M')|_{\mathcal{X}}$ , where  $M'$  uniquely determined, up to isomorphism, by an object  $X$  in  $\mathcal{X}$ .*

To have a better view on the injective covariant  $\mathcal{X}$ -modules, let  $\mathcal{T}_{\mathcal{X}}$  denote the subcategory of all left  $\Lambda$ -modules  $M$  such that there exists  $X \in \mathcal{X}$  with  $D(-, X) \simeq (-\otimes_{\Lambda} M)|_{\mathcal{X}}$ . If we let  $\mathcal{X} = \text{mod-}\Lambda$ , then Gruson and Jensen's result stated in 5.1.1 imply that  $\mathcal{T}_{\text{mod-}\Lambda} = \Lambda\text{-mod}$ . Moreover, it is easy to verify that  $\mathcal{T}_{\text{prj-}\Lambda} = \Lambda\text{-inj}$ .

We end this subsection by the following result, which is of independent interest.

**Proposition 5.1.6.** *Let  $\mathcal{X} = \text{Gprj-}\Lambda$  be the subcategory of Gorenstein projective  $\Lambda$ -modules. Then  $\mathcal{T}_{\text{Gprj-}\Lambda} = \Lambda\text{-Ginj}$ .*

*Proof.* Let  $P$  be a right  $\Lambda$ -module. Consider the natural transformation  $\Upsilon_{(P, -)} : -\otimes_{\Lambda} D(P) \rightarrow D\text{Hom}_{\Lambda}(-, P)$ , defined on a  $\Lambda$ -module  $M$  by

$$\Upsilon_{(P, M)} : M \otimes D(P) \rightarrow D\text{Hom}_{\Lambda}(M, P), \quad x \otimes f \mapsto (\varphi \mapsto f(\varphi(x)))$$

for  $x \in M$ ,  $f \in D(P) = \text{Hom}_{\Lambda}(P, E)$ , and  $\varphi \in \text{Hom}_{\Lambda}(M, P)$ . It is easily seen that  $\Upsilon_{(-, P)}$  is an equivalence if  $P$  is a finitely generated projective module. Now assume that  $G$  is a Gorenstein projective  $\Lambda$ -module. So we have an exact sequence  $0 \rightarrow G \rightarrow P^0 \rightarrow P^1$  with  $P^0$  and  $P^1$  projective. This in turn, induces the following commutative diagram

$$\begin{array}{ccccccc} -\otimes_{\Lambda} D(P^1) & \longrightarrow & -\otimes_{\Lambda} D(P^0) & \longrightarrow & -\otimes_{\Lambda} D(G) & \longrightarrow & 0 \\ \downarrow \Upsilon_{(-, P^1)} & & \downarrow \Upsilon_{(-, P^0)} & & \downarrow \Upsilon_{(-, G)} & & \\ D\text{Hom}_{\Lambda}(-, P^1) & \longrightarrow & D\text{Hom}_{\Lambda}(-, P^0) & \longrightarrow & D\text{Hom}_{\Lambda}(-, G) & \longrightarrow & 0 \end{array}$$

in  $\mathcal{X}\text{-mod}$ . But  $\Upsilon_{(-, P^1)}$  and  $\Upsilon_{(-, P^0)}$  are isomorphisms so is  $\Upsilon_{(-, G)}$ . This implies the result.  $\square$

**5.2. Existence of Recollement.** In this subsection, we will introduce two more functors  $\kappa$  and  $v$  so that together with  $t$  defined in 5.1.2, we construct the desired recollement.

**5.2.1.** Let us start by introducing  $\kappa : \Lambda\text{-mod} \rightarrow \mathcal{X}\text{-mod}$ . Pick a left  $\Lambda$ -module  $M$  and consider an injective copresentation  $0 \rightarrow M \rightarrow I^0 \xrightarrow{d} I^1$  of it. By duality  $D = \text{Hom}(-, E(R/J))$ , there exist a morphism  $\delta : P_1 \rightarrow P_0$  of projective right  $\Lambda$ -modules such that  $D(\delta) \simeq d$ . Hence we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & I^0 & \xrightarrow{d} & I^1 \\ & & & & \downarrow \wr & & \downarrow \wr \\ & & & & D(P_0) & \xrightarrow{D(\delta)} & D(P_1). \end{array}$$

Define  $\kappa(M)$  as

$$\kappa(M) = \text{Ker}((-\otimes_{\Lambda} D(P_0))|_{\mathcal{X}} \xrightarrow{(-\otimes d)|_{\mathcal{X}}} (-\otimes_{\Lambda} D(P_1))|_{\mathcal{X}}).$$

Note that since  $\mathcal{X}$  contains projective right  $\Lambda$ -modules, the sequence

$$0 \longrightarrow \kappa(M) \longrightarrow (-\otimes_{\Lambda} D(P_0))|_{\mathcal{X}} \xrightarrow{(-\otimes d)|_{\mathcal{X}}} (-\otimes_{\Lambda} D(P_1))|_{\mathcal{X}}.$$

is an injective copresentation of  $\kappa(M)$  in  $\mathcal{X}\text{-mod}$ . The map  $\kappa$  can be naturally defined on the morphisms, so we leave it to the readers.

**5.2.2.** We define a functor  $v : \mathcal{X}\text{-mod} \rightarrow \Lambda\text{-mod}$  as follows. Let  $F \in \mathcal{X}\text{-mod}$ . Consider injective copresentation

$$0 \rightarrow G \rightarrow (- \otimes_{\Lambda} D(X_0))|_{\mathcal{X}} \xrightarrow{d} (- \otimes_{\Lambda} D(X_1))|_{\mathcal{X}}$$

of  $F$ . By Lemma 5.1.3, there exists a unique morphism  $f : D(X_0) \rightarrow D(X_1)$  such that  $d = (- \otimes f)|_{\mathcal{X}}$ . We define the functor  $v : \mathcal{X}\text{-mod} \rightarrow \Lambda\text{-mod}$  by the attachment

$$v(F) := \text{Ker}(f : D(X_0) \rightarrow D(X_1)).$$

In a natural way,  $v$  can be defined on the morphisms.

**5.2.3.** We denote by  $\mathcal{X}\text{-mod}^0$  the subcategory of  $\mathcal{X}\text{-mod}$  consisting of all functors that vanish on projective right  $\Lambda$ -modules. By definition, it can be seen that  $\mathcal{X}\text{-mod}^0$  is the kernel of the functor  $v$  defined in 5.2.2.

Now we have enough ingredients to state the main theorem of this subsection.

**Theorem 5.2.4.** *Let  $\mathcal{X}$  be a functorially finite subcategory of  $\text{mod-}\Lambda$  consisting  $\text{prj-}\Lambda$ . Then, there exists the recollement*

$$\begin{array}{ccccc} & \xleftarrow{j_{\lambda}} & & \xleftarrow{t} & \\ \mathcal{X}\text{-mod}^0 & \xrightarrow{j} & \mathcal{X}\text{-mod} & \xrightarrow{v} & \Lambda\text{-mod} \\ & \xleftarrow{j_{\rho}} & & \xleftarrow{\kappa} & \end{array}$$

of abelian categories.

*Proof.* For the proof of the existence of the recollement, first it should be investigated that  $v$  is an exact functor, then verify that  $(t, v)$  and  $(v, \kappa)$  are adjoint pairs and finally show that  $\kappa$  is fully faithful. Since it is just a routine check similar to what is done for the proof of Theorem 3.7, we skip the proof.  $\square$

Two special cases are in order as the following two examples.

**Example 5.2.5.** In the above theorem, set  $\mathcal{X} = \text{mod-}\Lambda$ . Then we have the following recollement

$$\begin{array}{ccccc} & \xleftarrow{\quad} & & \xleftarrow{\quad} & \\ (\text{mod-}\Lambda)\text{-mod}^0 & \xrightarrow{\quad} & (\text{mod-}\Lambda)\text{-mod} & \xrightarrow{\quad} & \Lambda\text{-mod} \\ & \xleftarrow{\quad} & & \xleftarrow{\quad} & \end{array}$$

**Example 5.2.6.** If  $\Lambda$  is a Gorenstein algebra or more generally a virtually Gorenstein algebra, then  $\text{Gprj-}\Lambda$  is a contravariantly finite subcategory of  $\text{mod-}\Lambda$ . Moreover, since it is a resolving subcategory of  $\text{mod-}\Lambda$  [Ho, Theorem 2.5], i.e. contains all projectives and is closed with respect to extensions and kernel of epimorphisms. Then by a result of Krause and Solberg [KS],  $\text{Gprj-}\Lambda$  is also covariantly finite and hence is a functorially finite subcategory of  $\text{mod-}\Lambda$ . Hence Theorem 5.2.4 applies and so we have the following recollement

$$\begin{array}{ccccc} & \xleftarrow{\quad} & & \xleftarrow{\quad} & \\ \text{Gprj-mod}^0 & \xrightarrow{\quad} & \text{Gprj-mod} & \xrightarrow{\quad} & \Lambda\text{-mod} \\ & \xleftarrow{\quad} & & \xleftarrow{\quad} & \end{array}$$

**Remark 5.2.7.** Assume that  $\Lambda$  is a self-injective artin algebra and  $\mathcal{X}$  is a functorially finite subcategory of  $\text{mod-}\Lambda$  containing  $\text{prj-}\Lambda$ . Then the recollement of Theorem 5.2.4 is the same as the recollement that is constructed in Theorem 3.8. To see this, just one should note that since  $\Lambda$  is self-injective,  $\text{prj-}\Lambda = \text{inj-}\Lambda$ .

**5.3. Dualities of the categories of right and left  $\Lambda$ -modules.** In this short subsection, associated to any functorially finite subcategory  $\mathcal{X} \supseteq \text{prj-}\Lambda$  of  $\text{mod-}\Lambda$ , a duality will be constructed between the categories of right and left  $\Lambda$ -modules, also in the stable level.

Let  $D : \text{mod-}\mathcal{X} \rightarrow \mathcal{X}\text{-mod}$  be the usual duality, that exists because  $\mathcal{X}$  is a dualising  $R$ -variety. It follows from the definition that  $D$  can be restricted to a functor

$$D|_{\text{mod}_0\text{-}\mathcal{X}} : \text{mod}_0\text{-}\mathcal{X} \rightarrow \mathcal{X}\text{-mod}^0.$$

Therefore, by Theorems 3.7 and 5.2.4 we get the following commutative diagram of abelian categories

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{mod}_0\text{-}\mathcal{X} & \longrightarrow & \text{mod-}\mathcal{X} & \longrightarrow & \frac{\text{mod-}\mathcal{X}}{\text{mod}_0\text{-}\mathcal{X}} \longrightarrow 0 \\ & & \downarrow D| & & \downarrow D & & \downarrow \overline{D} \\ 0 & \longrightarrow & \mathcal{X}\text{-mod}^0 & \longrightarrow & \mathcal{X}\text{-mod} & \longrightarrow & \frac{\mathcal{X}\text{-mod}}{\mathcal{X}\text{-mod}^0} \longrightarrow 0. \end{array}$$

such that the horizontal maps are duality. Now we can define the duality  $\tilde{D}_{\mathcal{X}}$  with respect to the subcategory  $\mathcal{X}$  as composition of the following functors

$$\text{mod-}\Lambda \xrightarrow{\overline{\vartheta}^{-1}} \frac{\text{mod-}\mathcal{X}}{\text{mod}_0\text{-}\mathcal{X}} \xrightarrow{\overline{D}} \frac{\mathcal{X}\text{-mod}}{\mathcal{X}\text{-mod}^0} \xrightarrow{\overline{v}} \Lambda\text{-mod},$$

where  $\overline{\vartheta}$  and  $\overline{v}$  induced from the functors  $\vartheta$  and  $v$  introduced in Theorems 3.7 and 5.2.4, respectively.

Now let  $P$  be a projective right  $\Lambda$ -module. Then

$$\tilde{D}_{\mathcal{X}}(P) = \overline{v}\overline{D}\overline{\vartheta}^{-1}(P) = \overline{v}\overline{D}((- , P)|_{\mathcal{X}}) = \overline{v}(D(- , P)) = \overline{\vartheta}(- \otimes_{\Lambda} D(P)) = D(P).$$

Hence right projective  $\Lambda$ -modules project to the left injective  $\Lambda$ -modules. Therefore the constructed duality functor  $\tilde{D}_{\mathcal{X}}$  induces a duality between  $\underline{\text{mod-}}\Lambda$  and  $\overline{\Lambda\text{-mod}}$ .

In particular, if  $\mathcal{X} = \text{prj-}\Lambda$ , then  $\tilde{D}_{\text{prj-}\Lambda}$ , provides the usual duality between the stable categories of right and left modules.

## 6. CATEGORICAL RESOLUTIONS OF BOUNDED DERIVED CATEGORIES

In this section, we show that  $\mathbb{D}^b(\text{mod-}\Lambda)$ , the bounded derived category of  $\Lambda$ , admits a categorical resolution, where  $\Lambda$  is an arbitrary artin algebra.

We begin by the definition of a categorical resolution of the bounded derived category of an artin algebra. Although, the definition in literature is for arbitrary triangulated categories, in this paper we only concentrate on the bounded derived categories of artin algebras. We follow the definition presented by [Z, Definition 1.1], which is a combination of a definition due to Bondal and Orlov [BO] and also another one due to Kuznetsov [Ku, Definition 3.2], both as different attempts for providing a categorical translation of the notion of the resolutions of singularities.

**Convention.** Throughout the section,  $\Lambda$  is an artin algebra and  $\mathcal{X}$  is a contravariantly finite subcategory of  $\text{mod-}\Lambda$  containing  $\text{prj-}\Lambda$ . For a subcategory  $\mathcal{B}$  of an abelian category  $\mathcal{A}$ ,  $\mathbb{C}(\mathcal{B})$ , resp.  $\mathbb{K}(\mathcal{B})$ , denote the category of complexes, resp. the homotopy category of complexes, over  $\mathcal{B}$ . Their full subcategories consisting of bounded complexes will be denoted by  $\mathbb{C}^b(\mathcal{B})$ , resp.  $\mathbb{K}^b(\mathcal{B})$ .

**Definition 6.1.** ([Z, Definition 1.1]) Let  $\Lambda$  be an artin algebra of infinite global dimension. A categorical resolution of  $\mathbb{D}^b(\text{mod-}\Lambda)$ , is a triple  $(\mathbb{D}^b(\text{mod-}\Lambda'), \pi_*, \pi^*)$ , where  $\Lambda'$  is an artin

algebra of finite global dimension and  $\pi_* : \mathbb{D}^b(\text{mod-}\Lambda') \longrightarrow \mathbb{D}^b(\text{mod-}\Lambda)$  and  $\pi^* : \mathbb{K}^b(\text{prj-}\Lambda) \longrightarrow \mathbb{D}^b(\text{mod-}\Lambda')$  are triangle functors satisfying the following conditions.

- (i)  $\pi_*$  induces a triangle-equivalence  $\frac{\mathbb{D}^b(\text{mod-}\Lambda')}{\text{Ker}\pi_*} \simeq \mathbb{D}^b(\text{mod-}\Lambda)$ ;
- (ii)  $(\pi^*, \pi_*)$  is an adjoint pair on  $\mathbb{K}^b(\text{prj-}\Lambda)$ . That is, for every  $\mathbf{P} \in \mathbb{K}^b(\text{prj-}\Lambda)$  and every  $\mathbf{X}' \in \mathbb{D}^b(\text{mod-}\Lambda')$ , there exists a functorial isomorphism

$$\mathbb{D}^b(\text{mod-}\Lambda')(\pi^*(\mathbf{P}), \mathbf{X}') \cong \mathbb{D}^b(\text{mod-}\Lambda)(\mathbf{P}, \pi_*(\mathbf{X}'));$$

- (iii) The unit  $\eta : 1_{\mathbb{K}^b(\text{prj-}\Lambda)} \longrightarrow \pi_*\pi^*$  is a natural isomorphism.

Furthermore, a categorical resolution  $(\mathbb{D}^b(\text{mod-}\Lambda'), \pi_*, \pi^*)$  of  $\mathbb{D}^b(\text{mod-}\Lambda)$  is called weakly crepant if  $\pi^*$  is also a right adjoint to  $\pi_*$  on  $\mathbb{K}^b(\text{prj-}\Lambda)$ .

**6.2. Definitions and Notations.** Let  $\Lambda$  and  $\mathcal{X}$  be as in our convention.

- (i) The exact functor  $\vartheta : \text{mod-}\mathcal{X} \longrightarrow \text{mod-}\Lambda$ , defined in Remark 3.2, can be extended naturally to  $\mathbb{D}^b(\text{mod-}\mathcal{X})$  to induce a triangle functor

$$\mathbb{D}_\vartheta^b : \mathbb{D}^b(\text{mod-}\mathcal{X}) \longrightarrow \mathbb{D}^b(\text{mod-}\Lambda).$$

It acts on objects, as well as roofs, terms by terms. Let us denote the kernel of  $\mathbb{D}_\vartheta^b$  by  $\mathbb{D}_0^b(\text{mod-}\mathcal{X})$ . By definition, it consists of all complexes  $\mathbf{K}$  such that  $\mathbb{D}_\vartheta^b(\mathbf{K}) \simeq 0$ . Clearly  $\mathbb{D}_0^b(\text{mod-}\mathcal{X})$  is a thick subcategory of  $\mathbb{D}^b(\text{mod-}\mathcal{X})$ . The induced functor

$$\mathbb{D}^b(\text{mod-}\mathcal{X})/\mathbb{D}_0^b(\text{mod-}\mathcal{X}) \longrightarrow \mathbb{D}^b(\text{mod-}\Lambda)$$

will be denoted by  $\widetilde{\mathbb{D}_\vartheta^b}$ .

- (ii) Let  $\mathbb{K}_{\Lambda\text{-ac}}^b(\text{mod-}\mathcal{X})$  denote the full subcategory of  $\mathbb{K}^b(\text{mod-}\mathcal{X})$  consisting of all complexes  $\mathbf{F}$  such that

$$\mathbf{F}(\Lambda) : \dots \longrightarrow F^{i-1}(\Lambda) \xrightarrow{\partial_\Lambda^{i-1}} F^i(\Lambda) \xrightarrow{\partial_\Lambda^i} F^{i+1}(\Lambda) \longrightarrow \dots,$$

is an acyclic complex of abelian groups. Note that if  $\mathbf{F}$  is a complex in  $\mathbb{K}_{\Lambda\text{-ac}}^b(\text{mod-}\mathcal{X})$ , then  $\mathbf{F}(P)$  is acyclic, for all  $P \in \text{prj-}\Lambda$ . The Verdier quotient  $\mathbb{K}^b(\text{mod-}\mathcal{X})/\mathbb{K}_{\Lambda\text{-ac}}^b(\text{mod-}\mathcal{X})$  will be denoted by  $\mathbb{D}_\Lambda^b(\text{mod-}\mathcal{X})$ .

The following proposition has been proved in [AAHV, Proposition 3.1.7] in slightly different settings. For the convenient of the reader, we provide a sketch of proof with some modifications to compatible it with our settings in this section.

**Proposition 6.3.** *Let  $\mathbf{F} \in \mathbb{C}(\text{mod-}\mathcal{X})$  be a complex over  $\text{mod-}\mathcal{X}$ . There exists an exact sequence*

$$0 \longrightarrow \mathbf{F}_0 \longrightarrow \mathbf{F} \longrightarrow (-, \mathbb{D}_\vartheta^b(\mathbf{F}))|_{\mathcal{X}} \longrightarrow \mathbf{F}_1 \longrightarrow 0,$$

where  $\mathbf{F}_0$  and  $\mathbf{F}_1$  are complexes over  $\text{mod}_0\text{-}\mathcal{X}$ .

*Proof.* Let  $\mathbf{F} = (\mathbf{F}, \partial^i)$ . By Remark 3.9, for every  $i \in \mathbb{Z}$ , there is an exact sequence

$$0 \longrightarrow F_0^i \longrightarrow F^i \longrightarrow (-, \vartheta(F^i))|_{\mathcal{X}} \longrightarrow F_1^i \longrightarrow 0,$$

such that  $F_0^i$  and  $F_1^i$  belong to  $\text{mod}_0\text{-}\mathcal{X}$ . In view of Remark 4.2.4, for every  $i \in \mathbb{Z}$ , there exists unique morphism  $\vartheta(\partial^i) : \vartheta(F^i) \longrightarrow \vartheta(F^{i+1})$  and hence unique morphisms  $\partial_0^i : F_0^i \longrightarrow F_0^{i+1}$  and

$\partial_1^i : F_1^i \longrightarrow F_1^{i+1}$ , making the following diagram commutative

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_0^i & \longrightarrow & F^i & \longrightarrow & (-, \vartheta(F^i))|_{\mathcal{X}} & \longrightarrow & F_1^i & \longrightarrow & 0 \\
& & \downarrow \partial_0^i & & \downarrow \partial^i & & \downarrow (-, \vartheta(\partial^i)) & & \downarrow \partial_1^i & & \\
0 & \longrightarrow & F_0^{i+1} & \longrightarrow & F^{i+1} & \longrightarrow & (-, \vartheta(F^{i+1}))|_{\mathcal{X}} & \longrightarrow & F_1^{i+1} & \longrightarrow & 0.
\end{array}$$

The uniqueness of  $\partial_0^i$ ,  $\partial_1^i$  and  $\vartheta(\partial^i)$  yield the existence of complexes  $\mathbf{F}_0$ ,  $\mathbf{F}_1$  and  $\mathbb{D}_{\vartheta}^b(\mathbf{F})$  that fits together to imply the result. For more details see the proof of Proposition 3.1.7 of [AAHV].  $\square$

Let  $\mathbb{K}_{\text{ac}}^b(\text{mod-}\mathcal{X})$  denote the full subcategory of  $\mathbb{K}^b(\text{mod-}\mathcal{X})$  consisting of all acyclic complexes. It is a thick triangulated subcategory of  $\mathbb{K}_{\Lambda\text{-ac}}^b(\text{mod-}\mathcal{X})$ . Consider the Verdier quotient

$$\mathbb{K}_{\Lambda\text{-ac}}^b(\text{mod-}\mathcal{X})/\mathbb{K}_{\text{ac}}^b(\text{mod-}\mathcal{X}).$$

Clearly, this quotient is a triangulated subcategory of  $\mathbb{D}^b(\text{mod-}\mathcal{X}) = \mathbb{K}^b(\text{mod-}\mathcal{X})/\mathbb{K}_{\text{ac}}^b(\text{mod-}\mathcal{X})$ .

**Corollary 6.4.** *With the assumptions as in our convention,*

$$\mathbb{D}_0^b(\text{mod-}\mathcal{X}) \simeq \mathbb{K}_{\Lambda\text{-ac}}^b(\text{mod-}\mathcal{X})/\mathbb{K}_{\text{ac}}^b(\text{mod-}\mathcal{X}).$$

*Proof.* Let  $\mathbf{F}$  be a complex in  $\mathbb{D}^b(\text{mod-}\mathcal{X})$ . For the proof, it is enough to show that if  $\mathbb{D}_{\vartheta}^b(\mathbf{F})$  is an acyclic complex, then  $\mathbf{F} \in \mathbb{K}_{\Lambda\text{-ac}}^b(\text{mod-}\mathcal{X})$ . But it follows from the exact sequence

$$0 \longrightarrow \mathbf{F}_0 \longrightarrow \mathbf{F} \longrightarrow (-, \mathbb{D}_{\vartheta}^b(\mathbf{F}))|_{\mathcal{X}} \longrightarrow \mathbf{F}_1 \longrightarrow 0,$$

of the above Proposition. The proof is hence complete.  $\square$

Let

$$\Psi : \mathbb{D}_{\Lambda}^b(\text{mod-}\mathcal{X}) = \frac{\mathbb{K}^b(\text{mod-}\mathcal{X})}{\mathbb{K}_{\Lambda\text{-ac}}^b(\text{mod-}\mathcal{X})} \longrightarrow \frac{\mathbb{K}^b(\text{mod-}\mathcal{X})/\mathbb{K}_{\text{ac}}^b(\text{mod-}\mathcal{X})}{\mathbb{K}_{\Lambda\text{-ac}}^b(\text{mod-}\mathcal{X})/\mathbb{K}_{\text{ac}}^b(\text{mod-}\mathcal{X})} = \frac{\mathbb{D}^b(\text{mod-}\mathcal{X})}{\mathbb{D}_0^b(\text{mod-}\mathcal{X})}$$

denote the equivalence of triangulated quotients [V2, Corollaire 4-3]. Clearly  $\Psi$  acts as identity on the objects but sends a roof  $\frac{f}{s}$  to the roof  $\frac{f/1}{s/1}$ .

The composition

$$\tilde{\vartheta} = \widetilde{\mathbb{D}_{\vartheta}^b} \Psi : \mathbb{D}_{\Lambda}^b(\text{mod-}\mathcal{X}) \longrightarrow \mathbb{D}^b(\text{mod-}\Lambda)$$

attaches to any complex  $\mathbf{F}$  the complex  $\tilde{\vartheta}(\mathbf{F})$ , where

$$\tilde{\vartheta}(\mathbf{F}) : \dots \longrightarrow \vartheta(F^{i-1}) \xrightarrow{\vartheta(\partial^{i-1})} \vartheta(F^i) \xrightarrow{\vartheta(\partial^i)} \vartheta(F^{i+1}) \longrightarrow \dots$$

Similarly,  $\tilde{\vartheta}$  sends a roof  $\mathbf{F} \xleftarrow{s} \mathbf{H} \xrightarrow{f} \mathbf{G}$  to the roof

$$\tilde{\vartheta}(\mathbf{F}) \xleftarrow{\tilde{\vartheta}(s)} \tilde{\vartheta}(\mathbf{H}) \xrightarrow{\tilde{\vartheta}(f)} \tilde{\vartheta}(\mathbf{G}),$$

where for each morphism  $f$  in  $\mathbb{K}^b(\text{mod-}\mathcal{X})$ ,  $\tilde{\vartheta}(f)$  is the homotopy equivalence of a chain map in  $\mathbb{C}^b(\text{mod-}\Lambda)$  obtained by applying  $\vartheta$  terms by terms on a chain map in the homotopy equivalence class of  $f$ .

**Proposition 6.5.** *The functor  $\tilde{\vartheta}$  is an equivalence of triangulated categories. In particular, the functor  $\widetilde{\mathbb{D}_{\vartheta}^b}$  is an equivalence of triangulated categories.*

*Proof.* It is obvious that  $\widetilde{\vartheta}$  is an equivalence if and only if  $\widetilde{\mathbb{D}}_{\vartheta}^b$  is so. This proves the second part. The proof of the first part, is just a modification of the proof of Proposition 3.1.9 of [AAHV] to our settings. So we leave it to the readers.  $\square$

**Remark 6.6.** The equivalence

$$\widetilde{\mathbb{D}}_{\vartheta}^b : \mathbb{D}^b(\text{mod-}\mathcal{X})/\mathbb{D}_0^b(\text{mod-}\mathcal{X}) \longrightarrow \mathbb{D}^b(\text{mod-}\Lambda).$$

is in fact a derived version of Auslander formula. This derived level formula has been proved by Krause [Krl] for the case where  $\mathcal{X} = \text{mod-}\Lambda$ .

To continue, we need an easy lemma.

**Lemma 6.7.** *Let  $\Lambda$  and  $\mathcal{X}$  be as in our convention. Let  $\mathbf{P} \in \mathbb{K}^b(\text{prj-}\Lambda)$  and  $\mathbf{G} \in \mathbb{K}^b(\text{mod}_0\text{-}\mathcal{X})$ . Then*

$$\mathbb{K}^b(\text{mod-}\mathcal{X})((-, \mathbf{P}), \mathbf{G}) = 0.$$

*Proof.* Let  $\mathbf{P} = (P^i, \partial_{\mathbf{P}}^i)$  and  $\mathbf{G} = (G^i, \partial_{\mathbf{G}}^i)$ . By Yoneda lemma, for any  $i \in \mathbb{Z}$ ,  $((-, P^i), G^i) \cong G^i(P^i)$ . But  $G^i(P^i) = 0$ , because  $G^i \in \text{mod}_0\text{-}\mathcal{X}$ . Hence, as we defined everything terms by terms, we deduce the results.  $\square$

**Remark 6.8.** Let  $\mathbf{F}$  be a complex in  $\mathbb{C}(\text{mod-}\mathcal{X})$ . By Proposition 6.3, there exists an exact sequence of complexes

$$0 \longrightarrow \mathbf{F}_0 \longrightarrow \mathbf{F} \longrightarrow (-, \mathbb{D}_{\vartheta}^b(\mathbf{F}))|_{\mathcal{X}} \longrightarrow \mathbf{F}_1 \longrightarrow 0.$$

such that  $\mathbf{F}_0$  and  $\mathbf{F}_1$  are complexes over  $\text{mod}_0\text{-}\mathcal{X}$ . This sequence can be divided to the following two short exact sequences of complexes

$$0 \longrightarrow \mathbf{F}_0 \longrightarrow \mathbf{F} \longrightarrow \mathbf{K} \longrightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathbf{K} \longrightarrow (-, \mathbb{D}_{\vartheta}^b(\mathbf{F})) \longrightarrow \mathbf{F}_1 \longrightarrow 0.$$

These two sequences, in turn, induce the following two triangles

$$\mathbf{F}_0 \longrightarrow \mathbf{F} \longrightarrow \mathbf{K} \rightsquigarrow \quad \text{and} \quad \mathbf{K} \longrightarrow (-, \mathbb{D}_{\vartheta}^b(\mathbf{F})) \longrightarrow \mathbf{F}_1 \rightsquigarrow,$$

in  $\mathbb{D}^b(\text{mod-}\mathcal{X})$ , where  $\mathbf{F}_0$  and  $\mathbf{F}_1$  are considered as objects of  $\mathbb{D}^b(\text{mod}_0\text{-}\mathcal{X})$ . We use these triangles in our next propositions.

**Remark 6.9.** The functor  $\vartheta_{\lambda} : \text{mod-}\Lambda \longrightarrow \text{mod-}\mathcal{X}$  defined in 3.4, attaches to each projective  $\Lambda$ -module  $P$  the projective functor  $(-, P)$  in  $\text{mod-}\mathcal{X}$ . Since  $\vartheta_{\lambda}$  is an additive functor, it can be extended to  $\mathbb{K}^b(\text{prj-}\Lambda)$  to induce a functor

$$\mathbb{K}_{\vartheta_{\lambda}}^b : \mathbb{K}^b(\text{prj-}\Lambda) \longrightarrow \mathbb{K}^b(\text{mod-}\mathcal{X}).$$

This functor maps a complex  $\mathbf{P} \in \mathbb{K}^b(\text{prj-}\Lambda)$  to the complex  $(-, \mathbf{P})$ . So in fact, it is a functor from  $\mathbb{K}^b(\text{prj-}\Lambda)$  to  $\mathbb{K}^b(\text{prj-}(\text{mod-}\mathcal{X}))$ , that is, for every complex  $\mathbf{P} \in \mathbb{K}^b(\text{prj-}\Lambda)$ ,  $\mathbb{K}_{\vartheta_{\lambda}}^b(\mathbf{P}) = (-, \mathbf{P})$  is a bounded complex of projective  $\mathcal{X}$ -modules.

**Proposition 6.10.** *Let  $\Lambda$  and  $\mathcal{X}$  be as above. Then for every complexes  $\mathbf{P} \in \mathbb{K}^b(\text{prj-}\Lambda)$  and  $\mathbf{F} \in \mathbb{D}^b(\text{mod-}\mathcal{X})$ , there exists an isomorphism*

$$\mathbb{D}^b(\text{mod-}\mathcal{X})(\mathbb{K}_{\vartheta_{\lambda}}^b(\mathbf{P}), \mathbf{F}) \cong \mathbb{D}^b(\text{mod-}\Lambda)(\mathbf{P}, \mathbb{D}_{\vartheta}^b(\mathbf{F})),$$

*of abelian groups. That is,  $\mathbb{K}_{\vartheta_{\lambda}}^b$  is left adjoint to  $\mathbb{D}_{\vartheta}^b$  on  $\mathbb{K}^b(\text{prj-}\Lambda)$ .*

*Proof.* Let  $\mathbf{F} \in \mathbb{D}^b(\text{mod-}\mathcal{X})$ . By Remark 6.8, there exists the following two triangles

$$\mathbf{F}_0 \longrightarrow \mathbf{F} \longrightarrow \mathbf{K} \rightsquigarrow, \quad \text{and} \quad \mathbf{K} \longrightarrow (-, \mathbb{D}_\vartheta^b(\mathbf{F})) \longrightarrow \mathbf{F}_1 \rightsquigarrow,$$

where  $\mathbf{F}_0$  and  $\mathbf{F}_1$  are objects of  $\mathbb{D}^b(\text{mod}_0\text{-}\mathcal{X})$ . Apply the functor  $\mathbb{D}^b(\text{mod-}\mathcal{X})(\mathbb{K}_{\vartheta_\lambda}^b(\mathbf{P}), -)$  on these triangles, induce the following two long exact sequences of abelian groups

$$(\mathbb{K}_{\vartheta_\lambda}^b(\mathbf{P}), \mathbf{F}_0) \longrightarrow (\mathbb{K}_{\vartheta_\lambda}^b(\mathbf{P}), \mathbf{F}) \longrightarrow (\mathbb{K}_{\vartheta_\lambda}^b(\mathbf{P}), \mathbf{K}) \longrightarrow (\mathbb{K}_{\vartheta_\lambda}^b(\mathbf{P}), \mathbf{F}_0[1])$$

and

$$(\mathbb{K}_{\vartheta_\lambda}^b(\mathbf{P}), \mathbf{F}_1[-1]) \longrightarrow (\mathbb{K}_{\vartheta_\lambda}^b(\mathbf{P}), \mathbf{K}) \longrightarrow (\mathbb{K}_{\vartheta_\lambda}^b(\mathbf{P}), (-, \mathbb{D}_\vartheta^b(\mathbf{F}))) \longrightarrow (\mathbb{K}_{\vartheta_\lambda}^b(\mathbf{P}), \mathbf{F}_1),$$

respectively, where all Hom groups are taken in  $\mathbb{D}^b(\text{mod-}\mathcal{X})$ . But since  $\mathbf{P} \in \mathbb{K}^b(\text{prj-}\Lambda)$ ,  $\mathbb{K}_{\vartheta_\lambda}^b(\mathbf{P}) = (-, \mathbf{P}) \in \mathbb{K}^b(\text{prj-}(\text{mod-}\mathcal{X}))$  and hence some known abstract facts in triangulated categories, e.g. [W, Corollary 10.4.7], apply to guarantee that all these Hom sets can be also considered in  $\mathbb{K}^b(\text{mod-}\mathcal{X})$ . This we do and so Lemma 6.7 now come to play to eventually establish the existence of the following isomorphism

$$\mathbb{K}^b(\text{mod-}\mathcal{X})(\mathbb{K}_{\vartheta_\lambda}^b(\mathbf{P}), \mathbf{F}) \cong \mathbb{K}^b(\text{mod-}\mathcal{X})(\mathbb{K}_{\vartheta_\lambda}^b(\mathbf{P}), (-, \mathbb{D}_\vartheta^b(\mathbf{F}))),$$

of abelian groups. Therefore, to complete the proof, it is enough to show that

$$\mathbb{K}^b(\text{mod-}\mathcal{X})(\mathbb{K}_{\vartheta_\lambda}^b(\mathbf{P}), (-, \mathbb{D}_\vartheta^b(\mathbf{F}))) \cong \mathbb{K}^b(\text{mod-}\Lambda)(\mathbf{P}, \mathbb{D}_\vartheta^b(\mathbf{F})).$$

This is a consequence of Yoneda lemma applying terms by terms in view of the fact that  $\mathbb{K}_{\vartheta_\lambda}^b(\mathbf{P}) = (-, \mathbf{P})$ . Note that since  $\mathbf{P}$  is a bounded complex of projectives, by [W, Corollary 10.4.7], the Hom set  $(\mathbf{P}, \mathbb{D}_\vartheta^b(\mathbf{F}))$  can be considered either in  $\mathbb{K}^b(\text{mod-}\mathcal{X})$  or in  $\mathbb{D}^b(\text{mod-}\mathcal{X})$ . The proof is now complete.  $\square$

Now we are in a position to state and prove the main theorem of this section. As it is mentioned in the introduction, it provides a generalization of a recent result due to Pu Zhang [Z, Theorem 4.1].

**Theorem 6.11.** *Let  $\Lambda$  be an artin algebra of infinite global dimension and  $\tilde{\Lambda}$  denote its  $A$ -algebra. Then  $(\mathbb{D}^b(\text{mod-}\tilde{\Lambda}), \mathbb{D}_\vartheta^b, \mathbb{K}_{\vartheta_\lambda}^b)$  is a categorical resolution of  $\mathbb{D}^b(\text{mod-}\Lambda)$ .*

*Proof.* Let  $n$  be the nilpotency index of  $\Lambda$ . By definition,  $\tilde{\Lambda} = \text{End}(M)$ , where  $M = \bigoplus_{1 \leq i \leq n} \frac{\Lambda}{J^i}$ . It is known that  $\tilde{\Lambda}$  is of finite global dimension. In fact, it is a quasi-hereditary algebra [DR]. Set  $\mathcal{X} := \text{add-}M$ . Then  $\text{mod-}\mathcal{X} \simeq \text{mod-}\tilde{\Lambda}$ . By Propositions 6.5 and 6.10, the triple  $(\mathbb{D}^b(\text{mod-}\tilde{\Lambda}), \mathbb{D}_\vartheta^b, \mathbb{K}_{\vartheta_\lambda}^b)$  satisfies the first two conditions of the Definition 6.1. So we only need to check condition (iii). This also trivially follows from the definition as  $\mathbb{D}_\vartheta^b \mathbb{K}_{\vartheta_\lambda}^b(\mathbf{P}) = \mathbb{D}_\vartheta^b((-, \mathbf{P})) = \mathbf{P}$ .  $\square$

Towards the end of the paper, we show that if  $\Lambda$  is a self-injective artin algebra of infinite global dimension, then the triple  $(\mathbb{D}^b(\text{mod-}\tilde{\Lambda}), \mathbb{D}_\vartheta^b, \mathbb{K}_{\vartheta_\lambda}^b)$  introduced in the above theorem, provides a weakly crepant categorical resolution of  $\mathbb{D}^b(\text{mod-}\Lambda)$ . To do this, we need some preparations. Let us begin by a lemma.

**Lemma 6.12.** *Let  $\Lambda$  and  $\mathcal{X}$  be as in our convention. Let  $I \in \text{inj-}\Lambda$ . Then the functor  $(-, I)|_{\mathcal{X}}$  is an injective object of  $\text{mod-}\mathcal{X}$ .*



*Proof.* Since  $\mathcal{X}$  is a contravariantly finite subcategory of  $\text{mod-}\Lambda$ , there exists an exact sequence  $X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} I \rightarrow 0$  of  $\Lambda$ -modules such that  $d_0$  and  $d_1$  are right  $\mathcal{X}$ -approximations of  $I$  and  $\text{Ker}d_0$ , respectively. This guarantees the existence of the exact sequence

$$(-, X_1) \xrightarrow{(-, d_1)} (-, X_0) \xrightarrow{(-, d_0)} (-, I)|_{\mathcal{X}} \rightarrow 0$$

in  $\text{mod-}\mathcal{X}$ . Hence  $(-, I)|_{\mathcal{X}}$  is a finitely presented functor. To show that it is injective, pick a short exact sequence  $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$  of  $\mathcal{X}$ -modules and apply the functor  $(-, (-, I)|_{\mathcal{X}})$  on it to get the sequence

$$0 \rightarrow (F'', (-, I)|_{\mathcal{X}}) \rightarrow (F, (-, I)|_{\mathcal{X}}) \rightarrow (F', (-, I)|_{\mathcal{X}}) \rightarrow 0.$$

Since by Proposition 3.6,  $(\vartheta, \vartheta_\rho)$  is an adjoint pair, we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (F'', (-, I)|_{\mathcal{X}}) & \longrightarrow & (F, (-, I)|_{\mathcal{X}}) & \longrightarrow & (F', (-, I)|_{\mathcal{X}}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (\vartheta(F''), I) & \longrightarrow & (\vartheta(F), I) & \longrightarrow & (\vartheta(F'), I) \longrightarrow 0, \end{array}$$

where the vertical arrows are isomorphisms. But, the lower row is exact, because  $I$  is an injective module and  $\vartheta$  is an exact functor by Lemma 3.3. Hence the upper row should be exact, that implies the result.  $\square$

**Remark 6.13.** Let  $\Lambda$  be a self-injective artin algebra. So  $\text{prj-}\Lambda = \text{inj-}\Lambda$ . Hence a complex  $\mathbf{P} \in \mathbb{K}^b(\text{prj-}\Lambda)$  is also a bounded complex of injectives. So by the above lemma,  $\mathbb{K}_{\vartheta_\lambda}^b(\mathbf{P}) = (-, \mathbf{P})$  is a complex of injective  $\mathcal{X}$ -modules. Therefore by [W, Corollary 10.4.7], all Hom sets with either  $\mathbf{P}$  or  $\mathbb{K}_{\vartheta_\lambda}^b(\mathbf{P})$  in the second variants, can be calculated either in  $\mathbb{D}^b(\text{mod-}\mathcal{X})$  or in  $\mathbb{K}^b(\text{mod-}\mathcal{X})$ .

**Lemma 6.14.** Let  $\Lambda$  and  $\mathcal{X}$  be as in our convention. Then for every complexes  $\mathbf{G} \in \mathbb{K}^b(\text{mod}_0\text{-}\mathcal{X})$  and  $\mathbf{M} \in \mathbb{K}^b(\text{mod-}\Lambda)$ ,

$$\mathbb{K}^b(\text{mod-}\mathcal{X})(\mathbf{G}, (-, \mathbf{M})|_{\mathcal{X}}) = 0.$$

*Proof.* Let  $\mathbf{G} = (G^i, \partial_{\mathbf{G}}^i)$  and  $\mathbf{M} = (M^i, \partial_{\mathbf{M}}^i)$ . Since, by Proposition 3.6,  $(\vartheta, \vartheta_\rho)$  is an adjoint pair, for every  $i \in \mathbb{Z}$ , we have an isomorphism

$$\text{mod-}\mathcal{X}(\mathbf{G}, (-, \mathbf{M})|_{\mathcal{X}}) \cong \text{mod-}\Lambda(\vartheta(\mathbf{G}), \mathbf{M}).$$

Hence  $\text{mod-}\mathcal{X}(\mathbf{G}, (-, \mathbf{M})|_{\mathcal{X}}) = 0$ , because  $\mathbf{G} \in \text{mod}_0\text{-}\mathcal{X} = \text{Ker}\vartheta$ . This can be extended naturally, terms by terms, to bounded complexes to prove the lemma.  $\square$

**Theorem 6.15.** Let  $\Lambda$  be a self-injective artin algebra of infinite global dimension. Then the triple  $(\mathbb{D}^b(\text{mod-}\tilde{\Lambda}), \mathbb{D}_{\vartheta}^b, \mathbb{K}_{\vartheta_\lambda}^b)$  introduced in Theorem 6.11, is a weakly crepant categorical resolution of  $\mathbb{D}^b(\text{mod-}\Lambda)$ .

*Proof.* We just should show that  $\mathbb{K}_{\vartheta_\lambda}^b$  is a right adjoint of  $\mathbb{D}_{\vartheta}^b$  on  $\mathbb{K}^b(\text{prj-}\Lambda)$ . Pick  $\mathbf{F} \in \mathbb{D}^b(\text{mod-}\mathcal{X})$  and  $\mathbf{P} \in \mathbb{K}^b(\text{prj-}\Lambda)$ . Use Remark 6.8, to deduce the existence of the following two triangles

$$\mathbf{F}_0 \rightarrow \mathbf{F} \rightarrow \mathbf{K} \rightsquigarrow, \quad \text{and} \quad \mathbf{K} \rightarrow (-, \mathbb{D}_{\vartheta}^b(\mathbf{F})) \rightarrow \mathbf{F}_1 \rightsquigarrow,$$

with  $\mathbf{F}_0$  and  $\mathbf{F}_1$  objects of  $\mathbb{D}^b(\text{mod}_0\text{-}\mathcal{X})$ . Apply the functor  $\mathbb{D}^b(\text{mod-}\mathcal{X})(-, \mathbb{K}_{\vartheta_\lambda}^b(\mathbf{P}))$  on these triangles to get two exact sequences of abelian groups. Apply Remark 6.13, to deduce that we may also compute the Hom sets in  $\mathbb{K}^b(\text{mod}_0\text{-}\mathcal{X})$ . Now we should use Lemma 6.14, to conclude the following isomorphism

$$\mathbb{K}^b(\text{mod-}\mathcal{X})(\mathbf{F}, \mathbb{K}_{\vartheta_\lambda}^b(\mathbf{P})) \cong \mathbb{K}^b(\text{mod-}\mathcal{X})((-, \mathbb{D}_{\vartheta}^b(\mathbf{F})), \mathbb{K}_{\vartheta_\lambda}^b(\mathbf{P})).$$

The extended version of Yoneda lemma finally helps us to establish the following isomorphism

$$\mathbb{K}^b(\text{mod-}\mathcal{X})((-, \mathbb{D}_{\vartheta}^b(\mathbf{F})), \mathbb{K}_{\vartheta\lambda}^b(\mathbf{P})) \cong \mathbb{K}^b(\text{mod-}\Lambda)(\mathbb{D}_{\vartheta}^b(\mathbf{F}), \mathbf{P})$$

of abelian groups. The proof is hence complete.  $\square$

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